A QUANTITATIVE SECOND ORDER MINIMALITY CRITERION FOR CAVITIES IN ELASTIC BODIES

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ABSTRACT. We consider a functional which models an elastic body with a cavity. We show that if a critical point has positive second variation then it is a strict local minimizer. We also provide a quantitative estimate.

1. INTRODUCTION

The role of roughness appearing onto the surfaces and interfaces of nano-structures has been proved to be of great significance in several fields such as micro-electronics, metallurgy and materials science. For instance the roughness can strongly modify the mechanical properties of multilayered structures as confirmed by the observation that dislocations, island and cracks can be generated from a rough surface (see [10]). Many efforts were devoted to the investigation on how to control the roughness appearing onto the surfaces and interfaces of nano-structures, leading to the study of the so called Drive Rearrangement Instability, i.e., the morphological surfaces instability of interfaces between solids generated by elastic stress. This phenomena has been detected, for instance, in hetero-epitaxial growth of thin films with a lattice mismatch between film and substrate and in stressed elastic solids with cavities.

The theoretical investigation of the stability of the free surface of a planar non-hydrostatically stressed solid has been performed in the pioneering papers by Asaro and Tiller [4] and Grinfeld [16]. These authors showed that the free surface is unstable with respect to a given family of sinusoidal fluctuation. They also gave a first insightful description of the phenomenon, nowadays named as Asaro-Grinfeld-Tiller instability, for which a thin film growing on a flat substrate remains flat up to a critical value of the thickness, after which, the free surface becomes unstable developing corrugations and irregularities. This instability is explained as a consequence of the presence of two competing energies, usually identified with a bulk elastic energy and a surface energy. After these results the interest of the scientific community on the rigorous mathematical study of the morphological instabilities has rapidly grown. Starting from the paper [17] where Grinfeld follows the Gibbs variational approach to model the morphology of thin films, it became clear that a second order variational analysis could be successfully used. This approach has been used in the context of epitaxial growth first for a one dimensional model in [6]. Then in [5] and [12] the model introduced in [17], which is a more realistic two-dimensional model, corresponding to three-dimensional configurations with planar symmetry, is studied and the problem of finding a proper functional setting is successfully addressed. This settled the framework in which a precise and detailed analysis of qualitative properties of regular equilibrium configurations has been carried out by Fusco and Morini in [14] via a second order variational analysis. Indeed they prove a sufficient condition for local minimality in terms of the positivity of second variation and provide a sufficiently complete picture of the phenomena that occur in epitaxially growing thin films.

Such detailed analysis is instead far from being complete in the framework of stressed elastic solids with cavities. In this paper we perform a second order variational analysis for a two-dimensional variational model that has been recently used to describe surface instability in morphological evolution of cavities in stressed solids (see for instance [15, 19, 21]) with the aim of deriving new minimality conditions for equilibria and studying their stability. The model can be

roughly described as follows. Consider a cavity in an elastic solid, that will be identified with a smooth compact set $F \subset \mathbb{R}^2$, starshaped with respect to the origin. The solid region is assumed to obey to the classical law of linear elasticity, so that the bulk energy can be written in the form

$$\int_{B_{R_0} \setminus F} Q(E(u)) \, dz,$$

where $E(u)$ is the symmetric gradient of the elastic displacement u and Q is a bilinear form depending on the material (see Section 2 for details). The surface energy is simply assumed to be the length of the boundary of F. Then the energy for a regular configuration is expressed by the functional

$$\mathcal{F}(F, u) := \int_{B_0 \setminus F} Q(E(u)) \, dz + \mathcal{H}^1(\partial F) \, .$$

In this framework the shape of the void plays a key role in the evolution of cavities in stressed solid bodies, while the effects of the volume changes are negligible. Hence, one usually assumes that the void evolves preserving its volume. The equilibria are therefore identified with minimizers of $\mathcal{F}(F, u)$ under the volume constraint $|F| = d$. Since admissible configurations need not to be regular, the energy of such configurations has to be defined via a relaxation procedure. This issue, together with the study of the regularity of minima, has been addressed (even for more general functional involving anisotropic surface energies) in [13] where, in order to keep track of the possible appearance of cracks, the relaxed functional with respect to the Hausdorff convergence has been studied. The relaxed functional can be expressed in the following form:

$$(1.1) \qquad \mathcal{F}(F, u) := \int_{B_0 \setminus F} Q(E(u)) \, dz + \mathcal{H}^1(\Gamma_F) + 2\mathcal{H}^1(\Sigma_F) \, ,$$

where F has finite perimeter, Γ_F is the "regular" part of ∂F and Σ_F represents the cracks (see Section 2).

The main result of the paper is a quantitative minimality criterion that relies on the study of the second variation of the functional (1.1). To be more precise we prove in Theorem 5.1 that if (F, u) is a smooth critical configuration and the non local quadratic form $\partial^2 \mathcal{F}(F, u)$ associated to the second variation of $\mathcal{F}$ at (F, u) is positively defined, then there exists a constant c_0 such that

$$(1.2) \qquad \mathcal{F}(G, v) \geq \mathcal{F}(F, u) + c_0 |G \Delta F|^2$$

for any given admissible configuration (G, v) sufficiently close to (F, u) with respect to the Hausdorff distance of sets. In particular this implies not only that (F, u) is a strict local minimizer of (1.1) but also provides a quantitative estimate of the deviation from minimality for configurations close to (F, u) in the spirit of the recent result obtained in [1]. The minimality criterion is then applied to the case of a disk subjected to radial stretching where the second variation can be explicitly estimated to prove the local and global minimality of the round configuration if the applied stress is sufficiently small.

We point out that an important open problem is how to remove the assumption of starshapedness. Indeed, even the explicit form of the relaxed functional is unknown.

We conclude by outlining the structure of the paper and making some comments about the proofs. In Section 3 we calculate the second variation of $\mathcal{F}$ at any regular configuration (see Theorem 3.3) and we exploit the volume constraint to define the associated quadratic form in a critical configuration. At the end of the section in Lemma 3.6 we prove a "weak" coercivity property of $\partial^2 \mathcal{F}(F, u)$ in a critical point, which is the first step towards the proof of Theorem 5.1. In Section 4, as an intermediate step, we prove that the positivity of the second variation implies the local minimality in the $C^{1,1}$-topology. The main point in achieving this result is to overcome the lack of $C^{1,1}$-coercivity, which would immediately imply the result. This is done by proving the stability of the weak coercivity under one parameter perturbation of the critical configuration (see Lemma 4.7). In section 5 we exploit the regularity theory for a class of obstacle problems which arise as perturbations of (1.1) to show that the $C^{1,1}$-minimality actually implies the minimality with respect to the Hausdorff distance thus proving the theorem. In the last section we apply the previous analysis to the explicit case of a disk subjected to a radial stretching.

2. Preliminaries

In this section we fix the notation and describe precisely the required background for our analysis. We are interested in cavities identified as closed sets F with $\mathcal{H}^1(\partial F) < +\infty$ and starshaped with respect to the origin. The fact that F is starshaped allows us to describe it as a subgraph of a function. Since F has finite perimeter, the function associated to its boundary turns out to have bounded pointwise total variation. This will allow us to deal with functions rather than sets.

We denote by $\mathbb{S}^1$ the unit circle in $\mathbb{R}^2$ and by $\sigma : \mathbb{R} \to \mathbb{S}^1$ the local diffeomorphism defined by $\sigma(\theta) = (\cos\theta, \sin\theta)$, by σ^{-1} its local inverse and by $\sigma^\perp(\theta) = (\sin\theta, -\cos\theta)$ its orthogonal. We set by $C_\#^2(\mathbb{R})$ the collection of functions in $C^2(\mathbb{R})$ which are 2π-periodic. In a similar way we shall define the function spaces $H_\#^1(\mathbb{R})$ etc.

With a slight abuse of notation we set

$$(2.1) \qquad BV_\#(\mathbb{R}) := \{g : \mathbb{R} \to [0, R_0] \mid g \text{ is u.s.c., } 2\pi\text{-periodic and } pV(g, [0, 2\pi]) < \infty\},$$

where $pV(g, [0, 2\pi])$ is the pointwise total variation of g in $[0, 2\pi]$ and R_0 is the radius of a large ball B_{R_0}. For a function $g \in BV_\#(\mathbb{R})$ we define the extended graph of g as $\Gamma_g \cup \Sigma_g$, where

$$(2.2) \qquad \Gamma_g := \{\rho\sigma(\theta) \in \mathbb{R}^2 \mid g^-(\theta) \le \rho \le g^+(\theta), \theta \in \mathbb{R}\}$$

and

$$(2.3) \qquad \Sigma_g = \overline{\{\rho\sigma(\theta) \in \mathbb{R}^2 \mid g^+(\theta) < \rho < g(\theta), \theta \in \mathbb{R}\}}.$$

Here $g^-(\theta) := \liminf_{\tilde\theta \to \theta} g(\tilde\theta)$ and $g^+(\theta) := \limsup_{\tilde\theta \to \theta} g(\tilde\theta)$. We shall refer to Σ_g as the *set of cracks*.

Let us consider a compact set $F \subset \overline{B}_{R_0}$ starshaped with respect to the origin. Then, for $\sigma \in \mathbb{S}^1$, we can write

$$F = \{r\sigma(\theta) \in \mathbb{R}^2 \mid \theta \in \mathbb{R}, 0 \le r \le \rho_F(\theta)\},$$

where ρ_F is the *radial function* of F and is defined by

$$\rho_F(\theta) := \sup\{\rho \in \mathbb{R} \mid \rho\sigma(\theta) \in F\}.$$

It is clear that $\rho_F : \mathbb{R} \to [0, R_0]$ is upper semicontinuous. Moreover we have the following result, see [13, Lemmata 2.2 and 2.3].

Lemma 2.1. *Let $F \subset \bar{B}_{R_0}$ be a closed set starshaped with respect to the origin and let ρ_F be the radial function of F. Then*

$$\partial F = \Gamma_{\rho_F} \cup \Sigma_{\rho_F}.$$

Moreover $\mathcal{H}^1(\partial F) < +\infty$ if and only if ρ_F has finite pointwise total variation.

The previous lemma rigorously shows that we may use radial functions instead of sets. Hence, for $g \in BV_\#(\mathbb{R})$ we set

$$F_g := \{\rho\sigma(\theta) \in \mathbb{R}^2 \mid 0 \le \rho \le g(\theta)\} \quad \text{and} \quad \Omega_g := B_{R_0} \setminus F_g.$$

We may think at F_g as the void and at Ω_g as the elastic solid.

We can now define properly the space of admissible pairs. Given $u_0 \in C^\infty(\mathbb{R}^2 \setminus B_{R_0})$ we set

$$(2.4) \qquad X(u_0) = \{(g, v) \mid g \in BV_\#(\mathbb{R}), v \in H_{\mathrm{loc}}^1(\mathbb{R}^2 \setminus F_g; \mathbb{R}^2), v \equiv u_0 \text{ outside } B_{R_0}\},$$

and we shall use the notation $X(0)$ for $u_0 \equiv 0$. We define also the following subspaces of $X(u_0)$

$$(2.5) \qquad \begin{aligned} X_{\mathrm{Lip}}(u_0) &:= \{(g, v) \in X(u_0) \mid g \text{ is Lipschitz}\}, \\ X_{\mathrm{reg}}(u_0) &:= \{(g, v) \in X(u_0) \mid g \in C_\#^\infty(\mathbb{R}), v \in C^\infty(\bar\Omega_g)\}. \end{aligned}$$

We are now in position to give the proper definition of convergence in $X(u_0)$.

Definition 2.2. A sequence $(g_n, v_n) \subset X(u_0)$ is said to converge to (g, v) in $X(u_0)$ and we write $(g_n, v_n) \xrightarrow{X} (g, v)$ if
 (1) $\sup_{n \in \mathbb{N}} \mathcal{H}^1(\partial F_{g_n}) < +\infty$,
 (2) $\partial F_{g_n} \to \partial F_g$ in Hausdorff metric,
 (3) $v_n \rightharpoonup v$ weakly in $H^1(\omega; \mathbb{R}^2)$ for any open set ω compactly contained in $\mathbb{R}^2 \setminus F_g$.

In view of [13, Lemma 2.6], we see that $X(u_0)$ is closed under the convergence of Definition 2.2.

The *elastic energy density* is defined by $Q(E(u)) := \frac{1}{2}\mathbb{C}E(u) : E(u)$, where $\mathbb{C}$ is the fourth order tensor

$$\mathbb{C}\xi := \begin{pmatrix} (2\mu + \lambda)\xi_{11} + \lambda\xi_{22} & 2\mu\xi_{12} \\ 2\mu\xi_{12} & (2\mu + \lambda)\xi_{22} + \lambda\xi_{11} \end{pmatrix}$$

and $E(u)$ is the symmetric gradient of u

$$E(u) := \frac{1}{2}(Du + (Du)^T).$$

The constants μ, λ are called the *Lamé coefficients* and they are assumed to satisfy the following ellipticity conditions

$$\mu > 0 \quad \text{and} \quad \lambda > -\mu.$$

Since $Q(\xi) \geq \min\{\mu, \mu + \lambda\}|\xi|^2$ for every symmetric 2×2 matrix ξ, the above conditions guarantee that Q is coercive. We also set the ellipticity constant

$$\eta := \min\{\mu, \mu + \lambda\}.$$

For a pair $(g, v) \in X_{\text{Lip}}(u_0)$ we may write the value of the functional (1.1) as

$$\mathcal{F}(g, v) = \int_{\Omega_g} Q(E(v))\, dz + \mathcal{H}^1(\Gamma_g).$$

Since this functional is not lower semicontinuous with respect to the convergence in $X(u_0)$, in order to effectively address the minimization problem we consider the relaxed functional

$$\bar{\mathcal{F}}(g, v) = \inf\{\liminf_{n \to \infty} \mathcal{F}(g_n, v_n) \mid (g_n, v_n) \in X_{\text{Lip}}(u_0), (g_n, v_n) \xrightarrow{X} (g, v)\}.$$

The following integral representation of $\bar{\mathcal{F}}$ is proved in [13, Theorem 3.1], where the more general case of anisotropic surface energy is also considered.

Theorem 2.3. *Let $(g, v) \in X(u_0)$, then*

$$\bar{\mathcal{F}}(g, v) = \int_{\Omega_g} Q(E(v))\, dz + \mathcal{H}^1(\Gamma_g) + 2\mathcal{H}^1(\Sigma_g).$$

From now on we will always deal with the relaxed functional appearing in Theorem 2.3 and with abuse of notation we will denote it simply by $\mathcal{F}(g, v)$. The minimization problem can now be properly stated as

$$(2.6) \qquad \min\{\mathcal{F}(g, v) \mid (g, v) \in X(u_0), |\Omega_g| = d\}$$

for some given constant $d < |B_{R_0}|$. Existence of solutions of the problem (2.6) is then ensured by [13, Theorem 3.2].

Given $g \in BV_\#(\mathbb{R})$ there is one particular elastic displacement v which is the minimizer of the elastic energy $\int_{\Omega_g} Q(E(v))\, dz$ under the boundary condition $v \equiv u_0$ outside B_{R_0}. We call this map *the elastic equilibrium* associated to g. If $(h, u) \in X(u_0)$ solves (2.6) then u has to be the elastic equilibrium associated with h.

Assume now that a solution (h, u) belongs to $X_{\text{reg}}(u_0)$ and $h > 0$, then (h, u) satisfy the Euler-Lagrange equations

$$(2.7) \qquad \begin{cases} \text{div}\,\mathbb{C}(E(u)) = 0 & \text{in } \Omega_h \\ \mathbb{C}(E(u))[\nu] = 0 & \text{on } \Gamma_h \\ Q(E(u)) - k_h = \text{const.} & \text{on } \Gamma_h, \end{cases}$$

where k_h is the curvature of Γ_h. The first two equations are standard whereas the third one is the first variation of the functional (1.1). This motivates the following definition.

Definition 2.4. A pair $(h, u) \in X_{\text{reg}}(u_0)$ is said to be critical if it solves the equations (2.7).

At the end we remark that if (h, u) is a critical pair, then from the first two equations in (2.7) it follows that u is the elastic equilibrium associated to h. We also point out that in the definition of a critical point we only need to assume h to be smooth. Indeed, if we only assume $(h, u) \in X(u_0)$ and $h \in C^\infty(\mathbb{R})$, then it follows from the standard elliptic regularity theory (see [2]) that $u \in C^\infty(\bar{\Omega}_h)$.

Finally, we recall a version of the Korn's inequality which will be used throughout the paper, see e.g. [18].

Theorem 2.5 (Korn's inequality). *Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with smooth boundary and $v \in W^{1,2}(\Omega; \mathbb{R}^2)$. There exists a constant $C = C(\Omega)$ such that if*

$$\int_\Omega Dv \, dz = \int_\Omega Dv^T \, dz,$$

then

$$\int_\Omega |Dv|^2 \, dz \leq C \int_\Omega |E(v)|^2 \, dz.$$

Moreover if Ω is an annulus $A(R, r)$, $R > r$, the constant C depends only on the ratio r/R and $C \to 4$ as $r/R \to 0$.

3. Calculation of the second variation

The goal of this section is to calculate the second variation of the functional $\mathcal{F}$ at any point $(h, u) \in X_{\mathrm{reg}}(u_0)$, where u is the elastic equilibrium associated to h and $0 < h < R_0$, see formula (3.4). We then define a quadratic form for a critical pair (3.16) and give a definition of positiveness of the second variation, see Definition 3.5.

To this aim we will introduce the following notation. Given a 2π-periodic function $f : \mathbb{R} \to \mathbb{R}$ we will denote by $\underline{f} : \mathbb{R}^2 \setminus \{0\} \to \mathbb{R}^2$ the map

$$(3.1) \qquad \underline{f}(z) := f\left(\sigma^{-1}\left(\frac{z}{|z|}\right)\right)\frac{z}{|z|}.$$

For a parameter $s \in (-\varepsilon, \varepsilon)$ let $(h_s, u_s) \in X_{\mathrm{reg}}(u_0)$ be a smooth perturbation of (h, u), where u_s is the elastic equilibrium associated to h_s. By smooth perturbation we mean that the function $(s, \theta) \mapsto h_s(\theta)$ is smooth and $\lim_{s \to 0} ||h_s - h||_{C^2(\mathbb{R})} = 0$. Moreover we set $\dot{h}_s = \frac{\partial}{\partial s} h_s$, $\dot{u}_s = \frac{\partial}{\partial s} u_s$ and h'_s is the derivative with respect to θ. Notations $\dot{u}, \dot{h}$ means that we evaluate the time derivatives at $s = 0$. We explicitly point out that $\dot{h}$ and $\dot{u}$ depend on h_s. Finally, for a given h, we define the set of functions

$$(3.2) \qquad \mathcal{A}(\Omega_h) := \{w : \Omega_h \to \mathbb{R}^2 \mid (h, w) \in X(0)\}.$$

Roughly this means that $w \in \mathcal{A}(\Omega_h)$ if $w = 0$ outside B_{R_0}.

We will first write the equation for $\dot{u}$. In the following we will denote by τ the tangent unit vector to Γ_h clockwise oriented and by ν the unit normal to Γ_h pointing outward the set F_h.

Proposition 3.1. *Let $(h, u) \in X_{\mathrm{reg}}(u_0)$ be such that u is the elastic equilibrium associated to h and $0 < h < R_0$. Suppose (h_s, u_s) is a smooth perturbation of (h, u). Then the function $\dot{u} \in \mathcal{A}(\Omega_h)$ satisfies*

$$(3.3) \qquad \begin{aligned} \int_{\Omega_h} \mathbb{C}E(\dot{u}) : E(w) \, dz &= \int_{\Gamma_h} \langle \underline{\dot{h}}, \nu \rangle \, \mathbb{C}E(u) : E(w) \, d\mathcal{H}^1 \\ &= -\int_{\Gamma_h} \mathrm{div}_\tau\left(\langle \underline{\dot{h}}, \nu \rangle \, \mathbb{C}E(u)\right) \cdot w \, d\mathcal{H}^1, \end{aligned}$$

for all $w \in \mathcal{A}(\Omega_h)$.

Proof. The proof is very similar to the one in [14]. Arguing as in [7, Proposition 8.1] we obtain a one parameter family of C^∞-diffeomorphisms $\Phi_s(\cdot) : \mathbb{R}^2 \setminus \{0\} \to \mathbb{R}^2 \setminus \{0\}$ such that $\Phi_s(z) = \underline{h_s}$ on ∂F_h.

Suppose first that $w \in \mathcal{A}(\Omega_h) \cap C^\infty(\bar{\Omega}_h)$. We may extend w outside Ω_h such that $w \in \mathcal{A}(\Omega_{h_s}) \cap C^\infty(\bar{\Omega}_{h_s})$. Since u_s is the elastic equilibrium in Ω_{h_s} we have

$$\int_{\Omega_{h_s}} \mathbb{C}E(u_s) : E(w)\, dz = 0.$$

Differentiate this with respect to s and evaluate at $s = 0$ to obtain

$$\int_{\Omega_h} \mathbb{C}E(\dot{u}) : E(w)\, dz - \int_0^{2\pi} \dot{h}\, h\, [\mathbb{C}E(u) : E(w)](h\, \sigma(\theta))\, d\theta = 0.$$

Using the area formula and notation (3.1) we may write

$$\int_{\Omega_h} \mathbb{C}E(\dot{u}) : E(w)\, dz = \int_{\Gamma_h} \langle \underline{\dot{h}}, \nu \rangle\, \mathbb{C}E(u) : E(w)\, d\mathcal{H}^1,$$

where we have used the fact that the normal can be written in polar coordinates as $\nu = \frac{h\sigma + h'\sigma^\perp}{\sqrt{h^2 + h'^2}}$. The rest will follow by integration by parts and from the fact that $\mathbb{C}E(u)[\nu] = 0$ on Γ_h as in (2.7).

To obtain (3.3) for every $w \in \mathcal{A}(\Omega_h)$ one may use a standard approximation argument. $\qquad\square$

Remark 3.2. Notice that the equality (3.3) clearly holds also for test functions of the form $\tilde{w}(z) = w(z) + Az + b$, where $w \in \mathcal{A}(\Omega_h)$, $b \in \mathbb{R}^2$ and A is an antisymmetric matrix.

In the next theorem we derive the formula for the second variation of $\mathcal{F}$.

Theorem 3.3. *Suppose that (h, u) and (h_s, u_s) are as in Proposition 3.1. Let ν be the outer normal of $\Gamma_h = \partial F_h$, τ be the tangent (positively oriented) of Γ_h and k be the curvature of Γ_h. The second variation of $\mathcal{F}$ at (h, u) is*

(3.4)
$$\begin{aligned}
\frac{d^2}{ds^2}\mathcal{F}(h_s, u_s)\big|_{s=0} = &-\int_{\Omega_h} 2Q(E(\dot{u}))\, dz + \int_{\Gamma_h} |\partial_\tau \langle \underline{\dot{h}}, \nu \rangle|^2\, d\mathcal{H}^1 \\
&- \int_{\Gamma_h} (\partial_\nu Q(E(u)) + k^2)\, \langle \underline{\dot{h}}, \nu \rangle^2\, d\mathcal{H}^1 \\
&+ \int_{\Gamma_h} (Q(E(u)) - k)\, \partial_\tau \left(\langle \underline{\dot{h}}, \nu \rangle \langle \underline{\dot{h}}, \tau \rangle \right)\, d\mathcal{H}^1 \\
&- \int_{\Gamma_h} (Q(E(u)) - k) \left(\frac{\langle \underline{\dot{h}}, \nu \rangle^2}{\langle \underline{h}, \nu \rangle} + \langle \underline{\ddot{h}}, \nu \rangle \right)\, d\mathcal{H}^1.
\end{aligned}$$

Proof. We will treat the elastic and the perimeter part separately and write

$$\mathcal{F}(h_s, u_s) = \int_{\Omega_{h_s}} Q(E(u_s))\, dz + \mathcal{H}^1(\Gamma_{h_s}) = \mathcal{F}_1(h_s, u_s) + \mathcal{F}_2(h_s).$$

Since h_s is smooth, we notice that $\Sigma_{h_s} = \emptyset$ and denote by Φ_s the family of diffeomorphisms as in the proof of Proposition 3.1.

1st Variation : We start by differentiating the perimeter part $\mathcal{F}_2(h_s)$.

Since $\mathcal{H}^1(\Gamma_{h_s}) = \int_0^{2\pi} \sqrt{h_s^2 + h_s'^2}\, d\theta$ we have

$$\frac{d}{ds}\mathcal{F}_2(h_s) = \int_0^{2\pi} \frac{h_s\, \dot{h}_s + h_s'\, \dot{h}_s'}{\sqrt{h_s^2 + h_s'^2}}\, d\theta.$$

Integrate the second term by parts and obtain

$$\int_0^{2\pi} \frac{h_s'\, \dot{h}_s'}{\sqrt{h_s^2 + h_s'^2}}\, d\theta = -\int_0^{2\pi} \left(\frac{h_s''}{\sqrt{h_s^2 + h_s'^2}} - \frac{h_s(h_s')^2 + (h_s')^2 h_s''}{(h_s^2 + h_s'^2)^{3/2}} \right) \dot{h}_s\, d\theta.$$

Then we have

(3.5)
$$\begin{aligned}
\frac{d}{ds}\mathcal{F}_2(h_s) &= \int_0^{2\pi} h_s\, \dot{h}_s \left(\frac{h_s^2 + 2h_s'^2 - h_s h_s''}{(h_s^2 + h_s'^2)^{3/2}} \right) d\theta = \int_0^{2\pi} h_s\, \dot{h}_s\, k_s(h_s\sigma)\, d\theta, \\
&= \int_{\Gamma_{h_s}} \langle \underline{\dot{h}_s}, \nu_{h_s} \rangle\, k_s\, d\mathcal{H}^1.
\end{aligned}$$

where $k_s = \frac{h_s^2 + 2h_s'^2 - h_s h_s''}{(h_s^2 + h_s'^2)^{3/2}}$ is the curvature of Γ_{h_s} in polar coordinates.

Let us now treat the elastic part $\mathcal{F}_1(h_s, u_s)$. Differentiate it with respect to s and get, as in the proof of Proposition 3.1,

$$\frac{d}{ds}\mathcal{F}_1(h_s, u_s) = \int_{\Omega_{h_s}} \mathbb{C}E(\dot{u}_s) : E(u_s)\, dz - \int_0^{2\pi} \dot{h}_s\, h_s\, Q(E(u_s))(h_s\sigma)\, d\theta.$$

The first term disappears since u_s satisfies the Euler-Lagrange equations (2.7) and $\dot{u}_s \in \mathcal{A}(\Omega_{h_s})$. Hence, we are only left with

$$(3.6) \qquad \frac{d}{ds}\mathcal{F}_1(h_s, u_s) = -\int_0^{2\pi} \dot{h}_s\, h_s\, Q(E(u_s))(h_s\sigma)\, d\theta = -\int_{\Gamma_{h_s}} \langle \dot{\underline{h}}_s, \nu_{h_s}\rangle\, Q(E(u_s))\, d\mathcal{H}^1.$$

Combining (3.5) and (3.6) gives the first variation of $\mathcal{F}$.

2nd Variation: We will divide the proof in two steps.

Step 1: As in [14], we begin by making a couple of general observations.

Let d be the signed distance function from Γ_h, i.e.,

$$d(z) := \begin{cases} -\mathrm{dist}(z, \Gamma_h) & \text{if } z \in F_h, \\ \mathrm{dist}(z, \Gamma_h) & \text{if } z \notin F_h. \end{cases}$$

Since the boundary Γ_h is a graph of a C^∞-function, d is C^1 function in a small tubular neighbourhood of Γ_{h_t}. Setting $\nu(z) := \nabla d(z)$ and $k(z) := (\mathrm{div}\,\nu)(z)$, we observe that on Γ_h, ν is the outer normal to Γ_h and k is the curvature of Γ_h.

First we claim that

$$(3.7) \qquad \partial_\nu k = -k^2, \quad \text{on } \Gamma_h.$$

Differentiating the identity $|\nu| = 1$ with respect to ν yields $D\nu[\nu] = 0$. This shows that

$$(3.8) \qquad D\nu = D_\tau \nu = k\tau \otimes \tau \quad \text{and} \quad \mathrm{div}\,\nu = \mathrm{div}_\tau \nu, \quad \text{on } \Gamma_h.$$

Differentiating the identity $D\nu[\nu] = 0$ yields $\sum_{j=1}^2 (\partial_{jk}^2 \nu_i \nu_j + \partial_j \nu_i \partial_k \nu_j) = 0$ for $k, i = 1, 2$. Hence we have

$$(\partial_\nu(D\nu))_{ik} = \sum_{j=1}^2 \partial_{jk}^2 \nu_i \nu_j = -\sum_{j=1}^2 \partial_j \nu_i \partial_k \nu_j = -\left((D\nu)^2\right)_{ik}$$

for $i, k = 1, 2$. Using the previous identity we obtain

$$\partial_\nu k = \mathrm{Trace}\left(\partial_\nu(D\nu)\right) = -\mathrm{Trace}\left((D\nu)^2\right) = -k^2 \qquad \text{on } \Gamma_h,$$

where the last equality follows from (3.8). Hence we have (3.7).

Next we claim that

$$(3.9) \qquad \langle \dot{\nu}, \tau\rangle = -\partial_\tau \langle \dot{\underline{h}}, \nu\rangle, \quad \text{on } \Gamma_h.$$

Recall that $\Phi_s(z) : \mathbb{R}^2 \setminus \{0\} \to \mathbb{R}^2 \setminus \{0\}$ is a one-parameter family of C^∞-diffeomorphisms such that $\Phi_s(z) = \underline{h_s}$ on Γ_h. Notice that we have

$$(3.10) \qquad \langle \dot{\Phi}, \nu\rangle = \langle \dot{\underline{h}}, \nu\rangle, \quad \text{on } \Gamma_h.$$

Differentiating $D\Phi_s^{-T} D\Phi_s^T[\nu] = \nu$ and calculating at $s = 0$ gives $D\dot{\Phi}^{-T}[\nu] = -D\dot{\Phi}^T[\nu]$. Differentiate the identity

$$\nu_s \circ \Phi_s = \frac{D\Phi_s^{-T}[\nu]}{|D\Phi_s^{-T}[\nu]|}$$

with respect to s, evaluate at $s = 0$ and use the previous identity to obtain

$$(3.11) \qquad \dot{\nu} + D\nu[\dot{\Phi}] = -D\dot{\Phi}^T[\nu] + \langle D\dot{\Phi}^T[\nu], \nu\rangle\, \nu, \quad \text{on } \Gamma_h.$$

By (3.8) we have $D\nu = D_\tau \nu^T$ on Γ_h. Therefore, multiplying (3.11) by τ we obtain

$$\begin{aligned}
\langle \dot{\nu}, \tau\rangle &= -\langle D\dot{\Phi}^T[\nu], \tau\rangle - \langle D\nu[\dot{\Phi}], \tau\rangle \\
&= -\langle D\dot{\Phi}^T[\nu], \tau\rangle - \langle D\nu^T[\dot{\Phi}], \tau\rangle \\
&= \langle \left(-D\langle \dot{\Phi}, \nu\rangle\right), \tau\rangle = -\partial_\tau \langle \dot{\underline{h}}, \nu\rangle \quad \text{on } \Gamma_h
\end{aligned}$$

and (3.9) is proven.

Step 2: Let us start with the perimeter part and differentiate (3.5)

$$\frac{d^2}{ds^2}\mathcal{F}_2(h_s)\big|_{s=0} = \overbrace{\int_0^{2\pi} h\,\dot{h}\,\dot{k}(h\sigma)\,d\theta}^{A} + \overbrace{\int_0^{2\pi} h\,\dot{h}^2\,\partial_\sigma k(h\sigma)\,d\theta}^{B}$$
$$+ \int_0^{2\pi} \dot{h}^2\,k(h\sigma)\,d\theta + \int_0^{2\pi} h\,\ddot{h}\,k(h\sigma)\,d\theta.$$

For the term A we have that

$$A = \int_0^{2\pi} h\,\dot{h}\,\dot{k}(h\sigma)\,d\theta = \int_{\Gamma_h} \langle \underline{\dot{h}},\nu\rangle\,\dot{k}\,d\mathcal{H}^1 = \int_{\Gamma_h} \langle \underline{\dot{h}},\nu\rangle\,\mathrm{div}_\tau\,\dot{\nu}\,d\mathcal{H}^1$$
$$= -\int_{\Gamma_h} \langle \dot{\nu},\tau\rangle\,\partial_\tau\langle \underline{\dot{h}},\nu\rangle\,d\mathcal{H}^1 = \int_{\Gamma_h} |\partial_\tau\langle \underline{\dot{h}},\nu\rangle|^2\,d\mathcal{H}^1,$$

where we have used (3.9). For the term B, noticing that

$$\partial_\sigma k = \frac{h}{\sqrt{h^2+h'^2}}\,\partial_\nu k - \frac{h'}{\sqrt{h^2+h'^2}}\,\partial_\tau k \quad \text{and} \quad \tau = \frac{h\sigma^\perp - h'\sigma}{\sqrt{h^2+h'^2}},$$

we may write

$$B = \int_0^{2\pi} h\,\dot{h}^2\,\partial_\sigma k(h\sigma)\,d\theta = \int_{\Gamma_h} \langle \underline{\dot{h}},\nu\rangle^2\,\partial_\nu k\,d\mathcal{H}^1 + \int_{\Gamma_h} \langle \underline{\dot{h}},\nu\rangle\langle \underline{\dot{h}},\tau\rangle\,\partial_\tau k\,d\mathcal{H}^1$$
$$= -\int_{\Gamma_h} \langle \underline{\dot{h}},\nu\rangle^2\,k^2\,d\mathcal{H}^1 - \int_{\Gamma_h} k\,\partial_\tau\left(\langle \underline{\dot{h}},\nu\rangle\langle \underline{\dot{h}},\tau\rangle\right)d\mathcal{H}^1,$$

where we have used (3.7) and integration by parts. Hence, we have

(3.12)
$$\frac{d^2}{ds^2}\mathcal{F}_2(h_s)\big|_{s=0} = \int_{\Gamma_h} |\partial_\tau\langle \underline{\dot{h}},\nu\rangle|^2\,d\mathcal{H}^1 - \int_{\Gamma_h} \langle \underline{\dot{h}},\nu\rangle^2\,k^2\,d\mathcal{H}^1$$
$$- \int_{\Gamma_h} k\,\partial_\tau\left(\langle \underline{\dot{h}},\nu\rangle\langle \underline{\dot{h}},\tau\rangle\right)d\mathcal{H}^1 + \int_{\Gamma_h} k\,\frac{\langle \underline{\dot{h}},\nu\rangle^2}{\langle \underline{h},\nu\rangle}\,d\mathcal{H}^1 + \int_{\Gamma_h} k\,\langle \underline{\ddot{h}},\nu\rangle\,d\mathcal{H}^1.$$

We are left with the elastic part. Differentiate (3.6) to obtain

$$\frac{d^2}{ds^2}\mathcal{F}_1(h_s,u_s)\big|_{s=0} = -\int_0^{2\pi} \mathbb{C}E(\dot{u}):E(u)\,h\,\dot{h}\,d\theta - \int_0^{2\pi} \partial_\sigma Q(E(u))\,h\,\dot{h}^2\,d\theta$$
$$- \int_0^{2\pi} Q(E(u))\,(\dot{h}^2 + h\ddot{h})\,d\theta.$$

Since $\dot{u}\in\mathcal{A}(\Omega_h)$, we may rewrite the first term using (3.3) as follows

$$\int_0^{2\pi} \mathbb{C}E(\dot{u}):E(u)\,h\,\dot{h}\,d\theta = \int_{\Gamma_h} \langle \underline{\dot{h}},\nu\rangle\,\mathbb{C}E(u):E(\dot{u})\,d\mathcal{H}^1$$
$$= \int_{\Omega_h} 2Q(E(\dot{u}))\,dz.$$

For the second term, noticing that

$$\partial_\sigma Q(E(u)) = \frac{h}{\sqrt{h^2+h'^2}}\,\partial_\nu Q(E(u)) - \frac{h'}{\sqrt{h^2+h'^2}}\,\partial_\tau Q(E(u))$$

and using integration by parts, we get

$$\int_0^{2\pi} \partial_\sigma Q(E(u))\,h\,\dot{h}^2\,d\theta = \int_{\Gamma_h} \partial_\nu Q(E(u))\,\langle \underline{\dot{h}},\nu\rangle^2\,d\mathcal{H}^1 + \int_{\Gamma_h} \partial_\tau Q(E(u))\left(\langle \underline{\dot{h}},\nu\rangle\langle \underline{\dot{h}},\tau\rangle\right)d\mathcal{H}^1$$
$$= \int_{\Gamma_h} \partial_\nu Q(E(u))\,\langle \underline{\dot{h}},\nu\rangle^2\,d\mathcal{H}^1 - \int_{\Gamma_h} Q(E(u))\,\partial_\tau\left(\langle \underline{\dot{h}},\nu\rangle\langle \underline{\dot{h}},\tau\rangle\right)d\mathcal{H}^1.$$

Finally we have that

$$
\begin{aligned}
\frac{d^2}{ds^2}\mathcal{F}_1(h_s, u_s)\big|_{s=0} = {} & -\int_{\Omega_h} 2Q(E(\dot{u}))\,dz - \int_{\Gamma_h} \partial_\nu Q(E(u))\,\langle \dot{\underline{h}}, \nu\rangle^2\,d\mathcal{H}^1 \\
& + \int_{\Gamma_h} Q(E(u))\,\partial_\tau\left(\langle \dot{\underline{h}}, \nu\rangle\langle \dot{\underline{h}}, \tau\rangle\right)d\mathcal{H}^1 - \int_{\Gamma_h} Q(E(u))\,\frac{\langle \dot{\underline{h}}, \nu\rangle^2}{\langle \underline{h}, \nu\rangle}\,d\mathcal{H}^1 \\
& - \int_{\Gamma_h} Q(E(u))\,\langle \ddot{\underline{h}}, \nu\rangle\,d\mathcal{H}^1.
\end{aligned}
\tag{3.13}
$$

Combining (3.13) with (3.12) yields the formula (3.4).

$\square$

In the formula (3.4) we considered any smooth perturbation h_s of h. However, in order to be admissible for our minimization problem, a perturbation h_s has to satisfy the volume constraint $|F_{h_s}| = |F_h|$, or equivalently

$$
\int_0^{2\pi} h_s^2\,d\theta = \int_0^{2\pi} h^2\,d\theta \qquad \text{for all } s > 0.
\tag{3.14}
$$

Remark 3.4. If $(h, u) \in X_{\mathrm{reg}}(u_0)$ is a critical pair and the perturbation (h_s) satisfies the volume constraint (3.14), then the last two terms in (3.4) vanish. Indeed one term vanishes because the term $Q(E(u)) - k$ is constant on Γ_h by (2.7). The second one vanishes since differentiating two times the volume constraint (3.14) with respect to s we obtain

$$
\int_{\Gamma_h} \frac{\langle \dot{\underline{h}}, \nu\rangle^2}{\langle \underline{h}, \nu\rangle} + \langle \ddot{\underline{h}}, \nu\rangle\,d\mathcal{H}^1 = 0.
$$

Motivated by the previous observation, for any $\psi \in H^1_\#(\mathbb{R})$ satisfying

$$
\int_0^{2\pi} h\,\psi\,d\theta = 0,
\tag{3.15}
$$

we define the quadratic form associated to a regular critical pair (h, u)

$$
\begin{aligned}
\partial^2 \mathcal{F}(h, u)[\psi] := {} & -\int_{\Omega_h} 2Q(E(u_\psi))\,dz + \int_{\Gamma_h} |\partial_\tau\langle \underline{\psi}, \nu\rangle|^2\,d\mathcal{H}^1 \\
& - \int_{\Gamma_h} \left(\partial_\nu Q(E(u)) + k^2\right)\langle \underline{\psi}, \nu\rangle^2\,d\mathcal{H}^1,
\end{aligned}
\tag{3.16}
$$

where $u_\psi \in \mathcal{A}(\Omega_h)$ is the unique solution to

$$
\int_{\Omega_h} \mathbb{C}E(u_\psi) : E(w)\,dz = -\int_{\Gamma_h} \mathrm{div}_\tau\left(\langle \underline{\psi}, \nu\rangle\,\mathbb{C}E(u)\right)\cdot w\,d\mathcal{H}^1, \qquad \forall w \in \mathcal{A}(\Omega_h).
\tag{3.17}
$$

We define now what we mean the second variation of $\mathcal{F}$ being positive at a critical pair.

Definition 3.5. Suppose that $(h, u) \in X_{\mathrm{reg}}(u_0)$ is a critical pair. The functional (1.1) has *positive second variation* at (h, u) if

$$
\partial^2 \mathcal{F}(h, u)[\psi] > 0
$$

for all $\psi \in H^1_\#(\mathbb{R})$ such that $\psi \neq 0$ and satisfies (3.15).

We point out that if the second variation is positive at a critical point (h, u), then the formula (3.4) and Remark 3.4 imply that for every smooth perturbation h_s of h satisfying the volume constraint $\frac{d^2}{ds^2}\mathcal{F}(h_s, u_s)\big|_{s=0} > 0$.

At the end of the section we prove the following compactness result.

Lemma 3.6. *Suppose that a critical pair* $(h, u) \in X_{\mathrm{reg}}(u_0)$ *is a point of positive second variation, and* $0 < h < R_0$. *Then there exists* $c_0 > 0$ *such that*

$$
\partial^2 \mathcal{F}(h, u)[\psi] \geq c_0 \|\langle \underline{\psi}, \nu\rangle\|^2_{H^1(\Gamma_h)},
$$

for every $\psi \in H^1_\#(\mathbb{R})$ *satisfying* (3.15).

Proof. First notice that the condition (3.15) can be written using the notation (3.1) as

$$\int_{\Gamma_h} \langle \underline{\psi}, \nu \rangle \, d\mathcal{H}^1 = 0. \tag{3.18}$$

Using the Sobolev-Poincaré inequality $||\langle \underline{\psi}, \nu \rangle||_{L^2(\Gamma_h)} \leq C ||\partial_\tau \langle \underline{\psi}, \nu \rangle||_{L^2(\Gamma_h)}$ and (3.18) we easily see that it suffices to show that

$$c_0 := \inf \left\{ \partial^2 \mathcal{F}(h,u)[\psi] \mid \psi \in H^1_\#(\mathbb{R}) \text{ satisfying (3.15)}, \int_{\Gamma_h} |\partial_\tau \langle \underline{\psi}, \nu \rangle|^2 \, d\mathcal{H}^1 = 1 \right\} > 0.$$

Choose a sequence (ψ_n) such that ψ_n are smooth, satisfy (3.15), $\int_{\Gamma_h} |\partial_\tau \langle \underline{\psi_n}, \nu \rangle|^2 \, d\mathcal{H}^1 = 1$ and

$$\partial^2 \mathcal{F}(h,u)[\psi_n] \to c_0.$$

By restricting to a subsequences, we may assume that $\langle \underline{\psi_n}, \nu \rangle \rightharpoonup f$ weakly in $H^1(\Gamma_h)$. By defining

$$\psi(\theta) := \frac{f(h(\theta)\sigma(\theta))}{\langle \sigma, \nu \rangle} = \frac{f(h(\theta)\sigma(\theta))}{h(\theta)} \sqrt{h^2(\theta) + h'^2(\theta)}$$

we see that $f = \langle \underline{\psi}, \nu \rangle$, for some $\psi \in H^1_\#(\mathbb{R})$. Moreover since $\int_{\Gamma_h} f \, d\mathcal{H}^1 = 0$, the function ψ satisfies (3.15).

Next we prove that $\mathcal{F}(h,u)$ has the following lower semicontinuity property

$$\lim_{n \to \infty} \partial^2 \mathcal{F}(h,u)[\psi_n] \geq \partial^2 \mathcal{F}(h,u)[\psi]. \tag{3.19}$$

Indeed, since $\langle \underline{\psi_n}, \nu \rangle \rightharpoonup \langle \underline{\psi}, \nu \rangle$ weakly in $H^1(\Gamma_h)$ then $\langle \underline{\psi_n}, \nu \rangle \to \langle \underline{\psi}, \nu \rangle$ strongly in $L^2(\Gamma_h)$. Therefore we only need to check the convergence of the first term in (3.16).

First of all, the smoothness of ψ_n implies that u_{ψ_n} is smooth. Consider a domain $\tilde{\Omega}_h = B_{2R_0} \setminus F_h$ and a map $w_n(z) = u_{\psi_n}(z) + A_n z + b_n$, where A_n is an antisymmetric matrix and $b_n \in \mathbb{R}^2$. Notice that $w_n \in H^1(\tilde{\Omega}_h)$. By choosing A_n such that $\int_{\tilde{\Omega}_h} Dw_n \, dz = \int_{\tilde{\Omega}_h} Dw_n^T \, dz$ we have by Korn's inequality (2.5) that $||Dw_n||_{L^2(\tilde{\Omega}_h)} \leq C ||E(w_n)||_{L^2(\tilde{\Omega}_h)}$. Moreover, since $u_{\psi_n} \equiv 0$ outside B_{R_0}, we have $||E(w_n)||_{L^2(\tilde{\Omega}_h)} = ||E(u_{\psi_n})||_{L^2(\Omega_h)}$. By the Remark 3.2 we may use w_n as a test function in (3.17) and using Hölder's inequality and the trace theorem we get

$$\begin{aligned}
\int_{\Omega_h} 2Q(E(u_{\psi_n})) \, dz &= - \int_{\Gamma_h} \text{div}_\tau \left(\langle \underline{\psi_n}, \nu \rangle \, \mathbb{C}E(u) \right) \cdot w_n \, d\mathcal{H}^1 \\
&\leq ||\langle \underline{\psi_n}, \nu \rangle \, \mathbb{C}E(u)||_{H^1(\Gamma_h)} ||w_n||_{L^2(\Gamma_h)} \\
&\leq C ||\langle \underline{\psi_n}, \nu \rangle \, \mathbb{C}E(u)||_{H^1(\Omega_h)} ||Dw_n||_{L^2(\tilde{\Omega}_h)} \\
&\leq C ||\langle \underline{\psi_n}, \nu \rangle \, \mathbb{C}E(u)||_{H^1(\Omega_h)} ||E(u_{\psi_n})||_{L^2(\Omega_h)} .
\end{aligned} \tag{3.20}$$

Therefore

$$||Dw_n||_{L^2(\tilde{\Omega}_h)} \leq C ||E(u_{\psi_n})||_{L^2(\Omega_h)} \leq C .$$

However, since $u_{\psi_n} \equiv 0$ outside B_{R_0} we get

$$|B_{2R_0} \setminus B_{R_0}| \, |A_n|^2 = \int_{B_{2R_0} \setminus B_{R_0}} |E(w_n)|^2 \, dz \leq C.$$

This implies that the matrices A_n are bounded and therefore $||Du_{\psi_n}||_{L^2(\Omega_h)} \leq C$.

By the compactness of the trace operator we now have that $u_{\psi_n} \to u_\psi$ in $L^2(\Gamma_h)$ up to a subsequence. Use u_{ψ_n} as a test function in (3.17) to obtain

$$\begin{aligned}
\lim_{n \to \infty} \int_{\Omega_h} 2Q(E(u_{\psi_n})) \, dz &= - \lim_{n \to \infty} \int_{\Gamma_h} \text{div}_\tau \left(\langle \underline{\psi_n}, \nu \rangle \, \mathbb{C}E(u) \right) \cdot u_{\psi_n} \, d\mathcal{H}^1 \\
&= - \int_{\Gamma_h} \text{div}_\tau \left(\langle \underline{\psi}, \nu \rangle \, \mathbb{C}E(u) \right) \cdot u_\psi \, d\mathcal{H}^1 \\
&= \int_{\Omega_h} 2Q(E(u_\psi)) \, dz.
\end{aligned}$$

This proves (3.19).

The claim now follows since if $\psi \neq 0$, the lower semicontinuity (3.19) implies
$$c_0 = \lim_{n \to \infty} \partial^2 \mathcal{F}(h, u)[\psi_n] \geq \partial^2 \mathcal{F}(h, u)[\psi] > 0.$$

On the other hand if $\psi \equiv 0$ then the constraint $\int_{\Gamma_h} |\partial_\tau \langle \underline{\psi_n}, \nu \rangle|^2 \, d\mathcal{H}^1 = 1$ yields
$$c_0 = \lim_{n \to \infty} \partial^2 \mathcal{F}(h, u)[\psi_n] = 1.$$

$\square$

4. $C^{1,1}$-LOCAL MINIMALITY

In this section we perform a second order analysis of the functional (1.1) with respect to $C^{1,1}$-topology in a spirit of [11]. The main result is Proposition 4.1 where it is shown that a critical point $(h, u) \in X_{\mathrm{reg}}(u_0)$ with positive second variation is a strict local minimizer in the $C^{1,1}$-topology and the functional satisfies a growth estimate. We point out that, according to Lemma 3.6, the second variation at (h, u) is coercive with respect to a norm which is weaker than the $C^{1,1}$-norm. Therefore the local minimality does not follow directly from Lemma 3.6. The idea is to prove a coercivity bound in a whole $C^{1,1}$-neighborhood of the critical point, which is carried out in Lemma 4.7. The main difficulty is to control the bulk energy, that will be done by using regularity theory for linear elliptic systems. We prove the main result at the beginning without worrying about the technicalities. All the technical lemmata are proven later in the section.

Proposition 4.1. *Suppose that the critical pair $(h, u) \in X_{\mathrm{reg}}(u_0)$ is a point of positive second variation such that $0 < h < R_0$. There exists $\delta > 0$ such that for any admissible pair $(g, v) \in X(u_0)$ with $g \in C^{1,1}_{\#}(\mathbb{R})$, $||g||_{L^2([0,2\pi))} = ||h||_{L^2([0,2\pi))}$ and $||h - g||_{C^{1,1}(\mathbb{R})} \leq \delta$ we have*
$$\mathcal{F}(g, v) \geq \mathcal{F}(h, u) + c_1 ||h - g||^2_{L^2([0,2\pi))}.$$

Proof. Assume first that $g \in C^{\infty}_{\#}(\mathbb{R})$ and $||h - g||_{C^2(\mathbb{R})} \leq \delta$. By scaling we may assume that $||h||_{L^2([0,2\pi))} = \left(\int_0^{2\pi} h^2 d\theta \right)^{\frac{1}{2}} = 1$. We define
$$g_t := \frac{h + t(g - h)}{||h + t(g - h)||_{L^2}}$$
so that g_t satisfies the volume constraint, and set
$$f(t) := \mathcal{F}(g_t, v_t),$$
where v_t are the elastic equilibria associated to g_t. We calculate $\frac{d^2}{dt^2} \mathcal{F}(g_t, v_t)$ for every $t \in [0, 1)$ by applying the formula (3.4) to $(g_t)_s = g_{t+s}$ of g_t and get

(4.1)
$$\begin{aligned}
f''(t) = \frac{d^2}{dt^2} \mathcal{F}(g_t, v_t) = & -\int_{\Omega_{g_t}} 2Q(E(\dot{v}_t)) \, dz + \int_{\Gamma_{g_t}} |\partial_{\tau_t} \langle \underline{\dot{g}_t}, \nu_t \rangle|^2 \, d\mathcal{H}^1 \\
& - \int_{\Gamma_{g_t}} (\partial_{\nu_t} Q(E(v_t)) + k_t^2) \langle \underline{\dot{g}_t}, \nu_t \rangle^2 \, d\mathcal{H}^1 \\
& + \int_{\Gamma_{g_t}} (Q(E(v_t)) - k_t) \, \partial_{\tau_t} \left(\langle \underline{\dot{g}_t}, \nu_t \rangle \langle \underline{\dot{g}_t}, \tau_t \rangle \right) \, d\mathcal{H}^1 \\
& - \int_{\Gamma_{g_t}} (Q(E(v_t)) - k_t) \left(\frac{\langle \underline{\dot{g}_t}, \nu_t \rangle^2}{\langle \underline{g_t}, \nu_t \rangle} + \langle \underline{\ddot{g}_t}, \nu_t \rangle \right) \, d\mathcal{H}^1.
\end{aligned}$$

Here ν_t is the outer normal, τ_t the tangent, k_t the curvature of Γ_{g_t} and $\dot{v}_t$ is the unique solution to
$$\int_{\Omega_{g_t}} \mathbb{C}E(\dot{v}_t) : E(w) \, dz = -\int_{\Gamma_{g_t}} \mathrm{div}_{\tau_t} \left(\langle \underline{\dot{g}_t}, \nu_t \rangle \mathbb{C}E(v_t) \right) \cdot w \, d\mathcal{H}^1, \qquad \forall w \in \mathcal{A}(\Omega_{g_t}).$$
Remark 3.4 and Lemma 3.6 yield
$$f''(0) = \frac{d^2}{dt^2} \mathcal{F}(g_t, v_t)\Big|_{t=0} = \partial^2 \mathcal{F}(h, u)[\dot{g}] \geq c_0 ||\langle \underline{\dot{g}}, \nu \rangle||^2_{H^1(\Gamma_h)}.$$

It will be shown later in Lemma 4.7 that, when $\delta > 0$ is chosen to be small enough, the previous inequality implies

$$(4.2) \qquad f''(t) = \frac{d^2}{dt^2}\mathcal{F}(g_t, v_t) \geq \frac{c_0}{2}||\langle \dot{g_t}, \nu_t \rangle||^2_{H^1(\Gamma_{g_t})} \qquad \text{for all } t \in [0, 1).$$

It is now clear that $||\langle \dot{g_t}, \nu_t \rangle||^2_{H^1(\Gamma_{g_t})} \geq c\,||\dot{g_t}||^2_{L^2([0,2\pi))}$ holds for all $t \in [0, 1]$. Since $\int_0^{2\pi} g^2\, d\theta = \int_0^{2\pi} h^2\, d\theta$ we have $\int_0^{2\pi}(h - g)^2 d\theta = 2\int_0^{2\pi} h(h - g)\, d\theta$ and therefore

$$(4.3) \qquad ||\dot{g_t}||^2_{L^2([0,2\pi))} = \frac{1}{||h + t(g - h)||^4_{L^2}}\left(\int_0^{2\pi}(h - g)^2 d\theta - \frac{1}{4}\left(\int_0^{2\pi}(h - g)^2\, d\theta\right)^2\right).$$

Since $\int_0^{2\pi}(h - g)^2 d\theta$ is very small we obtain from (4.3) that

$$(4.4) \qquad\qquad ||\dot{g_t}||^2_{L^2([0,2\pi))} \geq \frac{1}{2}||h - g||^2_{L^2([0,2\pi))}.$$

From (4.2) and (4.4) we conclude that $f''(t) \geq \tilde{c}||h - g||^2_{L^2}$. Since (h, u) is a critical pair we have $f'(0) = 0$ and therefore

$$\mathcal{F}(g, v) - \mathcal{F}(h, u) = f(1) - f(0) = \int_0^1 (1 - t)f''(t)\, dt$$

$$\geq \tilde{c}\,||h - g||^2_{L^2([0,2\pi))}\int_0^1 (1 - t)\, dt$$

$$= \frac{\tilde{c}}{2}\,||h - g||^2_{L^2([0,2\pi))},$$

which proves the claim when g is smooth.

When $g \in C^{1,1,}_{\#}(\mathbb{R})$ the claim follows by using a standard approximation. $\square$

It remains to prove (4.2). The proof is based on a compactness argument and for that we have to study the continuity of the second variation formula (3.4). To control the boundary terms in (3.4) we need fractional Sobolev spaces whose definition and basic properties are recalled here. The function h is as in Proposition 4.1 and Γ_h is its graph.

Definition 4.2. For $0 < s < 1$ and $1 < p < \infty$ we define the fractional Sobolev space $W^{s,p}(\Gamma_h)$ as the set of those functions $v \in L^p(\Gamma_h)$ for which the Gagliardo seminorm is finite, i.e.

$$(4.5) \qquad [v]_{s,p;\,\Gamma_h} = \left(\int_{\Gamma_h}\int_{\Gamma_h}\frac{|v(z) - v(w)|^p}{|z - w|^{1+sp}}\, d\mathcal{H}^1(w)d\mathcal{H}^1(z)\right)^{1/p} < \infty.$$

The fractional Sobolev norm is defined as $||v||_{W^{s,p}(\Gamma_h)} := ||v||_{L^p(\Gamma_h)} + [v]_{s,p;\,\Gamma_h}$.

The space $W^{-s,p}(\Gamma_h)$ is the dual space of $W^{s,p}(\Gamma_h)$ and the dual norm of a function v is defined as

$$||v||_{W^{-s,p}(\Gamma_h)} := \sup\left\{\int_{\Gamma_h} vu\, d\mathcal{H}^1(z) \mid ||u||_{W^{s,p}(\partial F_h)} \leq 1\right\}.$$

We also use the notation $H^s(\Gamma_h)$ for $W^{s,2}(\Gamma_h)$ for $-1 < s < 1$ and the convention $W^{0,p}(\Gamma_h) := L^p(\Gamma_h)$. By Jensen's inequality we have the following classical embedding theorem.

Theorem 4.3. Let $-1 \leq t \leq s \leq 1$, $q \geq p$ such that $s - 1/p \geq t - 1/q$. Then there is a constant C depending on t, s, p, q and on the C^1-norm of h such that

$$||v||_{W^{t,q}(\Gamma_h)} \leq C||v||_{W^{s,p}(\Gamma_h)}.$$

We also have the following trace theorem.

Theorem 4.4. If $p > 1$ there exists a continuous linear operator $T : W^{1,p}(\Omega_h) \to W^{1-1/p,p}(\Gamma_h)$ such that $Tv = v|_{\Gamma_h}$ whenever v is continuous on $\bar{\Omega}_h$. The norm of T depends on C^1-norm of h and γ.

The next lemma will be used frequently.

Lemma 4.5. *Let* $-1 < s < 1$ *and suppose that* v *is a smooth function on* Γ_h . *Then the following hold.*

(i) *If* $a \in C^1(\Gamma_h)$ *then*

$$||av||_{W^{s,p}(\Gamma_h)} \leq C||a||_{C^1(\Gamma_h)}||v||_{W^{s,p}(\Gamma_h)},$$

where the constant C *depends on* p, s *and the* C^1*-norm of* h.

(ii) *If* $\Psi : \Gamma_h \to \Psi(\Gamma_h)$ *is a* C^1*-diffeomorphism, then*

$$||v \circ \Psi||_{W^{s,p}(\Psi(\Gamma_h))} \leq C||v||_{W^{s,p}(\Gamma_h)},$$

where the constant C *depends on* p, s *and the* C^1*-norms of* h, Ψ *and* Ψ^{-1}.

We will also need to control regularity of the elastic equilibrium. To this aim, the following elliptic estimate turns out to be useful, see [14, Lemma 4.1].

Lemma 4.6. *Suppose* $(g,v) \in X(0)$ *is such that* $\gamma \leq g \leq R_0 - \gamma$, $g \in C^2_\#(\mathbb{R})$ *and* $v \in \mathcal{A}(\Omega_g)$ *satisfies*

$$(4.6) \qquad \int_{\Omega_g} \mathbb{C}E(v) : E(w)\,dz = \int_{\Omega_g} f : E(w)\,dz \qquad \text{for every } w \in \mathcal{A}(\Omega_g),$$

where $f \in C^1(\bar{\Omega}_g; \mathbb{M}^{2\times 2})$. *Then for any* $p > 2$ *we have the following estimate*

$$(4.7) \begin{aligned} ||E(v)||_{W^{1,p}(\Omega_g;\mathbb{M}^{2\times 2})} &+ ||\nabla \mathbb{C}E(v)||_{H^{-\frac{1}{2}}(\Gamma_g;\mathbb{T})} \\ &\leq C\left(||E(v)||_{L^2(\Omega_g;\mathbb{M}^{2\times 2})} + ||f||_{C^1(\bar{\Omega}_g;\mathbb{M}^{2\times 2})}\right), \end{aligned}$$

where $\mathbb{T}$ *denotes the space of third order tensors and the constant* C *depends on* γ, p *and the* C^2*-norm of* g.

We are now in position to give the proof of the inequality (4.2). To control the bulk energy we use techniques developed in [14]. The main difference is that we use directly elliptic regularity rather than deal with eigenvalues of compact operators.

Lemma 4.7. *Suppose that a critical pair* $(h,u) \in X(u_0)$ *is a point of positive second variation with* $0 < h < R_0$ *and* $||h||_{L^2} = 1$. *Then there exists* $\delta > 0$ *such that for any admissible pair* $(g,v) \in X_{\text{reg}}(u_0)$ *with* $||g||_{L^2} = 1$ *and* $||h-g||_{C^2(\mathbb{R})} \leq \delta$ *we have for*

$$g_t = \frac{h + t(g-h)}{||h + t(g-h)||_{L^2}},$$

that

$$(4.8) \qquad \frac{d^2}{dt^2}\mathcal{F}(g_t, v_t) \geq \frac{c_0}{2}||\langle \underline{\dot{g}_t}, \nu_t\rangle||^2_{H^1(\Gamma_{g_t})} \qquad \text{for all } t \in [0,1],$$

where v_t *is the elastic equilibrium associated to* g_t. *The constant* c_0 *is from the Lemma 3.6.*

Proof. Choose $\gamma > 0$ such that $\gamma \leq h \leq R_0 - \gamma$. Suppose that the claim is not true and there are pairs $(g_n, v_n) \in X_{\text{reg}}(u_0)$ and $t_n \in [0,1]$ with

$$||h - g_n||_{C^2(\mathbb{R})} \to 0$$

for which the claim doesn't hold. Denoting

$$\dot{g}_n := \frac{\partial}{\partial t}\Big|_{t=t_n}\left(\frac{h + t(g_n - h)}{||h + t(g_n - h)||_{L^2}}\right) \quad \text{and} \quad \ddot{g}_n := \frac{\partial^2}{\partial t^2}\Big|_{t=t_n}\left(\frac{h + t(g_n - h)}{||h + t(g_n - h)||_{L^2}}\right)$$

this implies

$$(4.9) \qquad \lim_{n\to\infty} \frac{\mathcal{F}''(g_n, v_n)}{||\langle \underline{\dot{g}_n}, \nu_n\rangle||^2_{H^1(\Gamma_{g_n})}} \leq \frac{c_0}{2},$$

where by the notation $\mathcal{F}''(g_n, v_n)$ we mean

$$
\begin{aligned}
\mathcal{F}''(g_n, v_n) = &-\int_{\Omega_{g_n}} 2Q(E(\dot{v}_n))\, dz + \int_{\Gamma_{g_n}} |\partial_{\tau_n}\langle \underline{\dot{g}_n}, \nu_n\rangle|^2\, d\mathcal{H}^1 \\
&- \int_{\Gamma_{g_n}} (\partial_{\nu_n} Q(E(v_n)) + k_n^2)\, \langle \underline{\dot{g}_n}, \nu_n\rangle^2\, d\mathcal{H}^1 \\
&+ \int_{\Gamma_{g_n}} (Q(E(v_n)) - k_n)\, \partial_{\tau_n}\left(\langle \underline{\dot{g}_n}, \nu_n\rangle\langle \underline{\dot{g}_n}, \tau_n\rangle\right)\, d\mathcal{H}^1 \\
&- \int_{\Gamma_{g_n}} (Q(E(v_n)) - k_n)\left(\frac{\langle \underline{\dot{g}_n}, \nu_n\rangle^2}{\langle \underline{g_n}, \nu_n\rangle} + \langle \underline{\ddot{g}_n}, \nu_n\rangle\right)\, d\mathcal{H}^1 \\
= &\, I_1 + I_2 + I_3 + I_4 + I_5.
\end{aligned}
$$

(4.10)

Here ν_n is the outer normal to F_{g_n}, τ_n and k_n are the tangent vector and the curvature of Γ_{g_n} and $\dot{v}_n$ is the unique solution to

$$(4.11) \qquad \int_{\Omega_{g_n}} \mathbb{C}E(\dot{v}_n) : E(w)\, dz = -\int_{\Gamma_{g_n}} \mathrm{div}_{\tau_n}\left(\langle \underline{\dot{g}_n}, \nu_n\rangle\, \mathbb{C}E(v_n)\right)\cdot w\, d\mathcal{H}^1, \qquad \forall w \in \mathcal{A}(\Omega_{g_t}).$$

As in Proposition 3.1 we find C^∞-diffeomorphisms $\Psi_n : \bar{\Omega}_h \to \bar{\Omega}_{g_n}$ such that $\Psi_n : \Gamma_h \to \Gamma_{g_n}$ and

$$||\Psi_n - id||_{C^2(\bar{\Omega}_h; \mathbb{R}^2)} \leq C||h - g_n||_{C^2(\mathbb{R})}.$$

The goal is to examine the contribution of each term in (4.10) to the limit (4.9). We begin by proving that the contribution of I_4 and I_5 to (4.9) is zero.

Notice that the C^2-convergence of g_n implies $k_n \circ \Psi_n \to k$ in $L^\infty(\Gamma_h)$. Moreover, since v_n solves the first two equations in (2.7) and $\sup_n ||g_n||_{C^2([0,2\pi))} \leq C$, we have by a Schauder type estimate for Lamé system, see [14], that there is $\alpha \in (0,1)$ such that

$$(4.12) \qquad \sup_n ||v_n||_{C^{1,\alpha}(\bar{\Omega}'_n; \mathbb{R}^2)} < \infty, \qquad \text{for } \Omega'_n = B_{R_0 - \gamma} \setminus F_{g_n}.$$

Next we prove the following elliptic estimate

$$(4.13) \qquad \begin{aligned}
||E(u \circ \Psi_n^{-1}) - E(v_n)||_{W^{1,p}(\Omega_{g_n}; \mathbb{M}^{2\times 2})} &+ ||\nabla\mathbb{C}E(u \circ \Psi_n^{-1}) - \nabla\mathbb{C}E(v_n)||_{H^{-\frac{1}{2}}(\Gamma_{g_n}; \mathbb{T})} \\
&\leq C||h - g_n||_{C^2(\mathbb{R})},
\end{aligned}$$

where $p > 2$ and C depends on γ, p and the C^2-norms of h and u. Indeed by the equations (2.7) satisfied by u and v_n and a standard change of variables we obtain

$$(4.14) \qquad \int_{\Omega_{g_n}} \mathbb{C}(E(u \circ \Psi_n^{-1}) - E(v_n)) : E(w)\, dz = \int_{\Omega_{g_n}} f_n : E(w)\, dz, \qquad \forall w \in \mathcal{A}(\Omega_{g_n}),$$

where $f_n \in C^1(\bar{\Omega}_{g_n}; \mathbb{M}^{2\times 2})$ satisfies

$$||f_n||_{C^1(\bar{\Omega}_{g_n})} \leq C||h - g_n||_{C^2(\mathbb{R})}$$

for C depending only on the C^2-norm of u. Lemma 4.6 yields the estimate

$$
\begin{aligned}
||E(u \circ \Psi_n^{-1}) - E(v_n)||_{W^{1,p}(\Omega_{g_n}; \mathbb{M}^{2\times 2})} &+ ||\nabla\mathbb{C}E(u \circ \Psi_n^{-1}) - \nabla\mathbb{C}E(v_n)||_{H^{-\frac{1}{2}}(\Gamma_{g_n}; \mathbb{T})} \\
&\leq C\left(||E(u \circ \Psi_n^{-1}) - E(v_n)||_{L^2(\Omega_{g_n}; \mathbb{M}^{2\times 2})} + ||h - g_n||_{C^2(\mathbb{R})}\right).
\end{aligned}
$$

On the other hand, using $w = u \circ \Psi_n^{-1} - v_n$ as a test function in (4.14), we obtain

$$||E(u \circ \Psi_n^{-1}) - E(v_n)||_{L^2(\Omega_{g_n}; \mathbb{M}^{2\times 2})} \leq C||f_n||_{L^2(\Omega_{g_n}; \mathbb{M}^{2\times 2})}.$$

This concludes the proof of (4.13).

By the trace theorem 4.4, Lemma 4.5 and (4.13) we obtain

$$
\begin{aligned}
||E(v_n \circ \Psi_n) - E(u)||_{H^{\frac{1}{2}}(\Gamma_h; \mathbb{M}^{2\times 2})} &\leq C||E(v_n \circ \Psi_n) - E(u)||_{H^1(\Omega_h; \mathbb{M}^{2\times 2})} \\
&\leq C||g_n - h||_{C^2(\mathbb{R})}.
\end{aligned}
$$

This estimate together with (4.12) implies $v_n \circ \Psi_n \to u$ in $C^{1,\alpha}$. In particular, we have that

$$\left(Q(E(v_n)) - k_n\right) \circ \Psi_n \to Q(E(u)) - k \equiv \lambda$$

uniformly, where λ is a Lagrange multiplier. We may use this to estimate the term I_5 in (4.10). By explicit calculations one easily obtains that $||\langle \ddot{g}_n, \nu_n \rangle||_{L^1} \leq C ||\langle \dot{g}_n, \nu_n \rangle||^2_{L^2}$ and recalling that the functions $\dot{g}_n$ and $\ddot{g}_n$ satisfy the volume constraint, as in Remark 3.4,

$$\int_{\Gamma_{g_n}} \frac{\langle \dot{g}_n, \nu_n \rangle^2}{\langle \underline{g_n}, \nu_n \rangle} + \langle \ddot{g}_n, \nu_n \rangle \, d\mathcal{H}^1 = 0$$

we get

$$\int_{\Gamma_{g_n}} \left(Q(E(v_n)) - k_n \right) \left(\frac{\langle \dot{g}_n, \nu_n \rangle^2}{\langle \underline{g_n}, \nu_n \rangle} + \langle \ddot{g}_n, \nu_n \rangle \right) d\mathcal{H}^1$$

$$= \int_{\Gamma_{g_n}} \left(Q(E(v_n)) - k_n - \lambda \right) \left(\frac{\langle \dot{g}_n, \nu_n \rangle^2}{\langle \underline{g_n}, \nu_n \rangle} + \langle \ddot{g}_n, \nu_n \rangle \right) d\mathcal{H}^1$$

$$\leq C \, ||Q(E(v_n)) - k_n - \lambda||_{L^\infty(\Gamma_{g_n})} ||\langle \dot{g}_n, \nu_n \rangle||^2_{L^2(\Gamma_{g_n})}.$$

Using polar decomposition we have $\nu_n = \frac{g_n \sigma + g'_n \sigma^\perp}{\sqrt{g_n^2 + g_n'^2}}$ and $\tau_n = \frac{g_n \sigma^\perp - g'_n \sigma}{\sqrt{g_n^2 + g_n'^2}}$. Since $\langle \dot{g}_n, \tau_n \rangle(z) = \frac{\langle z, \tau_n \rangle}{\langle z, \nu_n \rangle} \langle \dot{g}_n, \nu_n \rangle(z)$ and $||\frac{\langle z, \tau_n \rangle}{\langle z, \nu_n \rangle}||_{H^1(\Gamma_{g_n})} \leq C$ we have as above that

$$\int_{\Gamma_{g_n}} \left(Q(E(v_n)) - k_n \right) \partial_{\tau_n} \left(\langle \dot{g}_n, \nu_n \rangle \langle \dot{g}_n, \tau_n \rangle \right) d\mathcal{H}^1$$

$$\leq C \, ||Q(E(v_n)) - k_n - \lambda||_{L^\infty(\Gamma_{g_n})} ||\langle \dot{g}_n, \nu_n \rangle||^2_{H^1(\Gamma_{g_n})}.$$

Hence the contribution of the terms I_4 and I_5 to the limit (4.9) is zero.

The remaining terms I_1, I_2 and I_3 form a quadratic form. The goal is to show that

(4.15) $$\lim_{n \to \infty} \frac{\mathcal{F}''(g_n, v_n)}{||\langle \dot{g}_n, \nu_n \rangle||^2_{H^1(\Gamma_{g_n})}} = \lim_{n \to \infty} \frac{\partial^2 \mathcal{F}(h, u)[\psi_n]}{||\langle \underline{\psi_n}, \nu \rangle||^2_{H^1(\Gamma_h)}}$$

where

$$\psi_n = \frac{\dot{g}_n g_n}{h}$$

and u_{ψ_n} solves

(4.16) $$\int_{\Omega_h} \mathbb{C} E(u_{\psi_n}) : E(w) \, dz = - \int_{\Gamma_h} \mathrm{div}_\tau \left(\langle \underline{\psi_n}, \nu \rangle \mathbb{C} E(u) \right) \cdot w \, d\mathcal{H}^1, \qquad \forall w \in \mathcal{A}(\Omega_h).$$

Notice that ψ_n satisfies the volume constraint $\int_0^{2\pi} h \psi_n \, d\theta = 0$ then we may use the Lemma 3.6 to conclude that

$$\lim_{n \to \infty} \frac{\partial^2 \mathcal{F}(h, u)[\psi_n]}{||\langle \underline{\psi_n}, \nu \rangle||^2_{H^1(\Gamma_h)}} \geq c_0,$$

which then contradicts (4.9) and proves the claim.

To show (4.15) we will compare the contribution of each terms in the quadratic form

$$\partial^2 \mathcal{F}(h, u)[\underline{\psi_n}] = - \int_{\Omega_h} 2 Q(E(u_{\psi_n})) \, dz + \int_{\Gamma_h} |\partial_\tau \langle \underline{\psi_n}, \nu \rangle|^2 \, d\mathcal{H}^1 - \int_{\Gamma_h} \left(\partial_\nu Q(E(u)) + k^2 \right) \langle \underline{\psi_n}, \nu \rangle^2 \, d\mathcal{H}^1$$

with respect to the one given by I_1, I_2 and I_3 in (4.10).

We first point out that since $g_n \to h$ in C^2 we have that $\nu_n \circ \Psi_n \to \nu$ and $\tau_n \circ \Psi_n \to \tau$ in $C^1(\Gamma_h)$. Therefore from the definition of ψ_n we get

$$\lim_{n \to \infty} \frac{||\langle \underline{\psi_n}, \nu \rangle||_{H^1(\Gamma_h)}}{||\langle \dot{g}_n, \nu_n \rangle||_{H^1(\Gamma_{g_n})}} = 1$$

and the convergence of I_2,

$$\lim_{n \to \infty} \frac{\int_{\Gamma_h} |\partial_\tau \langle \underline{\psi_n}, \nu \rangle|^2 \, d\mathcal{H}^1}{\int_{\Gamma_{g_n}} |\partial_{\tau_n} \langle \dot{g}_n, \nu_n \rangle|^2 \, d\mathcal{H}^1} = 1.$$

The convergence of I_1 follows from the equations (4.11) and (4.16). Indeed, by using standard change of variables, these equations yield

(4.17)
$$\int_{\Omega_{g_n}} \left(\mathbb{C}E(u_{\psi_n} \circ \Psi_n^{-1}) - \mathbb{C}E(\dot{v}_n)\right) : E(w)\, dz = \int_{\Omega_{g_n}} (\tilde{f}_n\, E(u_{\psi_n} \circ \Psi_n^{-1})) : E(w)\, dz + \int_{\Gamma_{g_n}} d_n \cdot w\, d\mathcal{H}^1$$

for any $w \in \mathcal{A}(\Omega_{g_n})$. Here

$$d_n = \mathrm{div}_\tau(\langle \underline{\psi_n}, \nu \rangle \mathbb{C}E(u)) \circ \Psi_n^{-1} |D_{\tau_n}\Psi_n^{-1}| - \mathrm{div}_{\tau_n}(\langle \underline{\dot{g}_n}, \nu_n \rangle \mathbb{C}E(v_n))$$

and $\tilde{f}_n \in L^2(\Omega_{g_n}; \mathbb{M}^{2 \times 2})$. For $\tilde{f}_n$ we have

(4.18) $$||\tilde{f}_n||_{L^\infty(\Omega_{g_n}; \mathbb{M}^{2 \times 2})} \to 0.$$

By the estimate (4.13) we get

$$||\nabla\mathbb{C}E(u \circ \Psi_n^{-1}) - \nabla\mathbb{C}E(v_n)||_{H^{-\frac{1}{2}}(\Gamma_{g_n}; \mathbb{T})} \to 0.$$

Therefore, by Lemma 4.5, the choice of Ψ_n and from $||\frac{\psi_n}{\dot{g}_n} - 1||_{C^1(\mathbb{R})} \to 0$ we have that

(4.19) $$||d_n||_{H^{-\frac{1}{2}}(\Gamma_{g_n}; \mathbb{R}^2)} ||\langle \underline{\dot{g}_n}, \nu_n \rangle||_{H^1(\Gamma_{g_n})}^{-1} \to 0.$$

Choose

$$w(z) = (u_{\psi_n} \circ \Psi_n^{-1} - \dot{v}_n)(z) + Az + b$$

as a test function in (4.17) where A is antisymmetric and b is a vector. This yields

(4.20)
$$\int_{\Omega_{g_n}} Q(E(u_{\psi_n} \circ \Psi_n^{-1} - \dot{v}_n))\, dz$$
$$\leq C||\tilde{f}_n||_{L^\infty(\Omega_{g_n}; \mathbb{M}^{2 \times 2})} ||E(u_{\psi_n})||_{L^2(\Omega_h; \mathbb{M}^{2 \times 2})} ||E(u_{\psi_n} \circ \Psi_n^{-1} - \dot{v}_n)||_{L^2(\Omega_{g_n}; \mathbb{M}^{2 \times 2})}$$
$$+ ||d_n||_{H^{-\frac{1}{2}}(\Gamma_{g_n}; \mathbb{R}^2)} ||w||_{H^{\frac{1}{2}}(\Gamma_{g_n}; \mathbb{R}^2)}.$$

By the Theorem 4.4 we get that

$$||w||_{H^{\frac{1}{2}}(\Gamma_{g_n}; \mathbb{R}^2)} \leq C||w||_{H^1(\Omega_{g_n}; \mathbb{M}^{2 \times 2})}.$$

As in the proof of Lemma 3.6 we choose A such that

$$||w||_{H^1(\Omega_{g_n}; \mathbb{M}^{2 \times 2})} \leq C||E(w)||_{L^2(\Omega_{g_n}; \mathbb{M}^{2 \times 2})} = C||E(u_{\psi_n} \circ \Psi_n^{-1} - \dot{v}_n)||_{L^2(\Omega_{g_n}; \mathbb{M}^{2 \times 2})},$$

by Korn's and Poincaré's inequalities (choose b accordingly). The two previous inequalities and (4.20) yield

$$||E(u_{\psi_n} \circ \Psi_n^{-1} - \dot{v}_n)||_{L^2(\Omega_{g_n}; \mathbb{M}^{2 \times 2})} \leq C\left(||\tilde{f}_n||_{L^\infty(\Omega_{g_n}; \mathbb{M}^{2 \times 2})} ||E(u_{\psi_n})||_{L^2(\Omega_h; \mathbb{M}^{2 \times 2})} + ||d_n||_{H^{-\frac{1}{2}}(\Gamma_{g_n}; \mathbb{R}^2)}\right).$$

Arguing as in (3.20) we may estimate

$$||E(u_{\psi_n})||_{L^2(\Omega_h; \mathbb{M}^{2 \times 2})} \leq C\, ||\langle \underline{\psi_n}, \nu \rangle||_{H^1(\Gamma_h)}.$$

Therefore using (4.18) and (4.19) we deduce that

$$\frac{||E(u_{\psi_n})||_{L^2(\Omega_h; \mathbb{M}^{2 \times 2})}^2 - ||E(\dot{v}_n)||_{L^2(\Omega_{g_n}; \mathbb{M}^{2 \times 2})}^2}{||\langle \underline{\dot{g}_n}, \nu_n \rangle||_{H^1(\Gamma_{g_n})}^2} \to 0.$$

This proves the convergence of I_1.

We are left with the term I_3 in (4.10). We need to show that

$$\left| \int_{\Gamma_{g_n}} (\partial_{\nu_n} Q(E(v_n)) + k_n^2)\, \langle \underline{\dot{g}_n}, \nu_n \rangle^2\, d\mathcal{H}^1 - \int_{\Gamma_h} (\partial_\nu Q(E(u)) + k^2)\, \langle \underline{\psi_n}, \nu \rangle^2\, d\mathcal{H}^1 \right| ||\langle \underline{\dot{g}_n}, \nu_n \rangle||_{H^1(\Gamma_{g_n})}^{-2} \to 0.$$

Due to the C^2-convergence of g_n and the C^1-convergence of $\frac{\psi_n}{g_n}$ we just need to show

(4.21) $$||\partial_{\nu_n} Q(E(v_n)) \circ \Psi_n - \partial_\nu Q(E(u))||_{H^{-\frac{1}{2}}(\Gamma_h)} \to 0.$$

This will be done as [14, Proposition 4.5]. For every $\varphi \in H^{\frac{1}{2}}(\Gamma_h)$ we have

$$\int_{\Gamma_h} \left(\frac{\partial}{\partial x_1} Q(E(v_n)) \circ \Psi_n - \frac{\partial}{\partial x_1} Q(E(u)) \right) \varphi \, d\mathcal{H}^1$$

$$= \int_{\Gamma_h} \left(\mathbb{C}E\left(\frac{\partial v_n}{\partial x_1}\right) \circ \Psi_n - \mathbb{C}E\left(\frac{\partial u}{\partial x_1}\right) \right) : (E(v_n) \circ \Psi_n) \, \varphi \, d\mathcal{H}^1$$

$$+ \int_{\Gamma_h} \mathbb{C}E\left(\frac{\partial u}{\partial x_1}\right) : (E(v_n) \circ \Psi_n - E(u)) \, \varphi \, d\mathcal{H}^1$$

$$\leq ||(\nabla \mathbb{C}E(v_n)) \circ \Psi_n - \nabla \mathbb{C}E(u)||_{H^{-\frac{1}{2}}(\Gamma_h;\mathbb{T})} ||(E(v_n) \circ \Psi_n)\varphi||_{H^{\frac{1}{2}}(\Gamma_h;\mathbb{M}^{2 \times 2}))}$$

$$+ C||E(v_n) \circ \Psi_n - E(u)||_{L^2(\Gamma_h;\mathbb{M}^{2 \times 2})} ||\varphi||_{L^2(\Gamma_h)}$$

where the constant depends on C^2-norms of u and h. Fix $p > 2$. By the definition of Gagliardo seminorm, Hölder's inequality, Theorem 4.3 and Theorem 4.4, we obtain

$$||(E(v_n) \circ \Psi_n)\varphi||_{H^{\frac{1}{2}}(\Gamma_h;\mathbb{M}^{2 \times 2}))}$$

$$\leq C||(E(v_n) \circ \Psi_n)||_{L^\infty(\Gamma_h;\mathbb{M}^{2 \times 2}))} ||\varphi||_{H^{\frac{1}{2}}(\Gamma_h)} + C||(E(v_n) \circ \Psi_n)||_{W^{\frac{p+2}{2p}, \frac{2p}{p-2}}(\Gamma_h;\mathbb{M}^{2 \times 2}))} ||\varphi||_{L^p(\Gamma_h)}$$

$$\leq C \left(||(E(v_n) \circ \Psi_n)||_{L^\infty(\Gamma_h;\mathbb{M}^{2 \times 2}))} + ||(E(v_n) \circ \Psi_n)||_{W^{1, \frac{2p}{p-2}}(\Omega_h;\mathbb{M}^{2 \times 2})} \right) ||\varphi||_{H^{\frac{1}{2}}(\Gamma_h)}.$$

By repeating the previous argument for $\frac{\partial}{\partial x_2}$ we obtain by (4.12) and (4.13) that

$$||\nabla Q(E(v_n)) \circ \Psi_n - \nabla Q(E(u))||_{H^{-\frac{1}{2}}(\Gamma_h;\mathbb{R}^2)}$$

(4.22)
$$\leq C \left(||(\nabla \mathbb{C}E(v_n)) \circ \Psi_n - \nabla \mathbb{C}E(u)||_{H^{-\frac{1}{2}}(\Gamma_h;\mathbb{T})} + ||E(v_n) \circ \Psi_n - E(u)||_{L^2(\Gamma_h;\mathbb{M}^{2 \times 2})} \right)$$

$$\leq C||g_n - h||_{C^2(\mathbb{R})}.$$

Since $\nu_n \circ \Psi_n \to \nu$ in C^1, (4.22) implies (4.21). This concludes the convergence of the term I_3 and completes the proof. $\square$

5. Local minimality

This section is devoted to prove the main result of the paper, the local minimality criterion. Namely, we show that if a critical point $(h, u) \in X_{\text{reg}}(u_0)$ has positive second variation, then it is a strict local minimizer in the Hausdorff distance of sets and a quantitative estimate in terms of the measure of the symmetric difference between the minimum and a competitor holds. Due to the sharp quantitative isoperimetric inequality, the exponent 2 in (5.1) is optimal.

Theorem 5.1. *Suppose that $(h, u) \in X_{\text{reg}}(u_0)$ is a critical pair for $\mathcal{F}$ with $0 < h < R_0$. If the second variation of $\mathcal{F}$ is positive at (h, u), then there is $\delta > 0$ such that for any $(g, v) \in X(u_0)$ with $|\Omega_g| = |\Omega_h|$ and $0 < d_{\mathcal{H}}(\Gamma_g \cup \Sigma_g, \Gamma_h) \leq \delta$ it holds*

(5.1)
$$\mathcal{F}(g, v) > \mathcal{F}(h, u) + c\,|\Omega_g \Delta \Omega_h|^2,$$

for some $c > 0$.

The proof is based on a contradiction argument and follows some ideas contained in [14], [9] and [1]. Assume that (h_n, u_n) is a contradicting sequence, i.e.,

$$\mathcal{F}(h_n, u_n) \leq \mathcal{F}(h, u) + c_0\,|\Omega_{h_n} \Delta \Omega_h|^2 \quad \text{and} \quad 0 < d_{\mathcal{H}}(\Gamma_{h_n} \cup \Sigma_{h_n}, \Gamma_h) \leq \frac{1}{n}.$$

The idea is to replace (h_n, u_n) by (g_n, v_n), which minimize an auxiliary constrained-penalized problem, and to prove that (g_n, v_n) are sufficiently regular to apply the $C^{1,1}$-minimality criterion to get a contradiction. As auxiliary problem we choose

$$\min \left\{ \mathcal{F}(g, v) + \Lambda \big||\Omega_g| - |\Omega_h|\big| + \sqrt{(|\Omega_g \Delta \Omega_h| - \varepsilon_n)^2 + \varepsilon_n} \; : \; (g, v) \in X(u_0) \,, \; g \leq h + \frac{1}{n} \right\},$$

where the second penalization term will provide the quantitative estimate in (5.1) and the obstacle $g \leq h + 1/n$ plays a key role in proving the regularity of (g_n, v_n).

The regularity proof is divided in three steps. In Lemma 5.6 we prove that g_n is Lipschitz using some geometrical arguments. Then, in Lemma 5.7, we show that g_n is a quasiminimizer for the area functional which in turns implies $C^{1,\alpha}$-regularity. Finally, we deduce the $C^{1,1}$-regularity in Lemma 5.8, by using the Euler-Lagrange equation for (g_n, v_n).

The following isoperimetric-type result will be used frequently in this section. The proof can be found in [1, Lemma 4.1].

Lemma 5.2.

(i) *Let $f \in C_\#^\infty(\mathbb{R})$ be non-negative and let $g \in BV_\#(\mathbb{R})$, then there exists a constant C, depending only on f, such that*

$$\mathcal{H}^1(\Gamma_g) - \mathcal{H}^1(\Gamma_f) \geq -C|\Omega_g \Delta \Omega_f|.$$

(ii) *Suppose D is a set of finite perimeter. Then*

$$P(D \cup B_r(x)) - P(B_r(x)) \geq \frac{1}{r}|D|,$$

where P stands for the perimeter.

We will also need the following property of concave functions.

Lemma 5.3. *Suppose that $f_n \in C^1(\mathbb{R})$ and $f \in C^1(\mathbb{R})$ are such that $||f_n - f||_{L^\infty(\mathbb{R})} \to 0$. If f_n are concave then it holds*

$$||f_n - f||_{C_{loc}^1(\mathbb{R})} \to 0.$$

Proof. Let $R > 0$ and fix $\varepsilon > 0$. Since $f \in C^1(\mathbb{R})$ we find $\delta > 0$ such that

$$f(\delta + x) - f(x) \geq f'(x)\delta - \varepsilon\delta$$

for every $|x| \leq R$. On the other hand, since f_n are concave, we have

$$\frac{f_n(\delta + x) - f_n(x)}{\delta} \leq f_n'(x).$$

Hence

$$f'(x) - f_n'(x) \leq \frac{f(\delta + x) - f_n(\delta + x) - (f(x) - f_n(x))}{\delta} + \varepsilon \leq 2\varepsilon,$$

when n is large. The reverse inequality $f_n'(x) - f'(x) \leq 2\varepsilon$ follows from a similar argument. □

We begin the study of the properties of solutions of the auxiliary problem by proving an exterior ball condition.
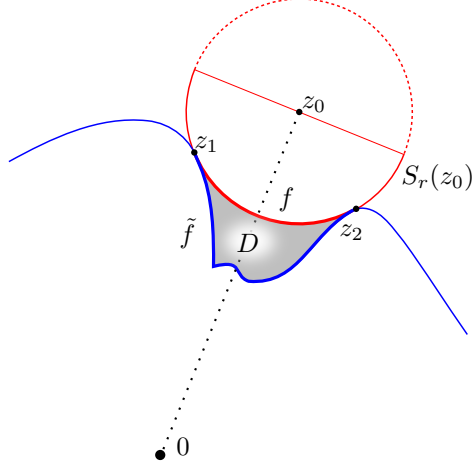
Theorem 5.4. *Let $h \in C_\#^\infty(\mathbb{R})$ such that $0 < h < R_0$ and $r < \min\{1/(\Lambda + 1), 1/\|\kappa_h\|_\infty\}$, where κ_h is the curvature of Γ_h. Then for every $c, \varepsilon \in [0,1]$ and $n \in \mathbb{N}$ any solution of the problem*

$$(5.2) \quad \min\left\{\mathcal{F}(g,v) + \Lambda\big||\Omega_g| - |\Omega_h|\big| + c\sqrt{(|\Omega_g \Delta \Omega_h| - \varepsilon)^2 + \varepsilon} : (g,v) \in X(u_0), g \leq h + \frac{1}{n}\right\},$$

satisfies the uniform exterior ball condition, *i.e., there exists $r > 0$ such that for every $z \in \partial F_g$ we find z_0 for which $B_r(z_0) \subset \mathbb{R}^2 \setminus F_g$ and $\partial B_r(z_0) \cap (\Gamma_g \cup \Sigma_g) = \{z\}$.*

Proof. The proof follows the argument from [14, Lemma 6.7]. Recall that $\partial F_g = \Gamma_g \cup \Sigma_g$. Given a ball $B_r(z_0)$ define the half circle $S_r(z_0) = \partial B_r(z_0) \cap \{z \in \mathbb{R}^2 : \langle z - z_0, z_0 \rangle < 0\}$. Assume that there exists a ball $B_r(z_0) \subset \mathbb{R}^2 \setminus F_g$ such that $S_r(z_0)$ intersects $\Gamma_g \cup \Sigma_g$ in two different points $z_1 = (\rho_1, \theta_1)$ and $z_2 = (\rho_2, \theta_2)$. When $r < 1/\|\kappa_h\|_\infty$ it is clear that the arc $f = f(\theta)$ of $S_r(z_0)$ connecting z_1 and z_2 satisfies $f(\theta) < h(\theta)$ for $\theta \in (\theta_1, \theta_2)$. Let $\tilde{g}$ be defined by $\tilde{g} = f$ for $\theta \in (\theta_1, \theta_2)$ and $\tilde{g} = g$ otherwise. Denote by $\tilde{f}$ the arc of $\Gamma_g \cup \Sigma_g$ connecting z_1 and z_2 and by D the region enclosed by $f \cup \tilde{f}$, see Figure 5.1.

FIGURE 5.1.

Notice that

$$
\sqrt{(|\Omega_{\tilde{g}}\Delta\Omega_h| - \varepsilon)^2 + \varepsilon} - \sqrt{(|\Omega_g\Delta\Omega_h| - \varepsilon)^2 + \varepsilon}
$$

(5.3)
$$
= \frac{(|\Omega_{\tilde{g}}\Delta\Omega_h| - \varepsilon)^2 - (|\Omega_g\Delta\Omega_h| - \varepsilon)^2}{\sqrt{(|\Omega_{\tilde{g}}\Delta\Omega_h| - \varepsilon)^2 + \varepsilon} + \sqrt{(|\Omega_g\Delta\Omega_h| - \varepsilon)^2 + \varepsilon}}
$$
$$
\leq \frac{(|\Omega_{\tilde{g}}\Delta\Omega_h| + |\Omega_g\Delta\Omega_h| - 2\varepsilon)(|\Omega_{\tilde{g}}\Delta\Omega_h| - |\Omega_g\Delta\Omega_h|)}{\big||\Omega_{\tilde{g}}\Delta\Omega_h| - \varepsilon\big| + \big||\Omega_g\Delta\Omega_h| - \varepsilon\big|}
$$
$$
\leq |\Omega_{\tilde{g}}\Delta\Omega_g| \, .
$$

Since $\Omega_{\tilde{g}}\Delta\Omega_g = D$ and $\Omega_{\tilde{g}} \subset \Omega_g$ we see that

(5.4)
$$
\mathcal{F}(\tilde{g}, v) + \Lambda\big||\Omega_{\tilde{g}}| - |\Omega_h|\big| + c\sqrt{(|\Omega_{\tilde{g}}\Delta\Omega_h| - \varepsilon)^2 + \varepsilon}
$$
$$
\leq \mathcal{F}(g, v) + \Lambda\big||\Omega_{\tilde{g}}| + |\Omega_h|\big| + c\sqrt{(|\Omega_g\Delta\Omega_h| - \varepsilon)^2 + \varepsilon} + \mathcal{H}^1(f) - \mathcal{H}(\tilde{f}) + (\Lambda + 1)|D| \, .
$$

Since $P(D \cup B_r(z_0)) - P(B_r(z_0)) \leq \mathcal{H}^1(f) - \mathcal{H}(\tilde{f})$, Lemma 5.2 implies

$$
\mathcal{H}^1(f) - \mathcal{H}(\tilde{f}) \leq -\frac{1}{r}|D| \, .
$$

Hence, since $r < 1/(\Lambda + 1)$, the inequality (5.4) contradicts the minimality of (g, v). The conclusion now follows arguing as [8, Lemma 2] or [12, Proposition 3.3, Step 2]. □

Lemma 5.5. *Let h, c, ε and n be as in the previous theorem. Suppose $(g, v) \in X(u_0)$ is any minimizer of (5.2). Then there exists $\Lambda_0 > 0$, independent of c, ε and n, such that if $\Lambda \geq \Lambda_0$ then $|\Omega_g| \geq |\Omega_h|$.*

Proof. We argue by contradiction supposing that $|\Omega_g| < |\Omega_h|$. We observe that there exists $0 < r < 1$ such that, if we define $\Omega_g^r = B_{R_0} \setminus rF_g$, we have $|\Omega_g^r| = |\Omega_h|$. Moreover, since

$$
|\Omega_g^r| = \pi R_0^2 - \frac{r^2}{2}\int_0^{2\pi} g^2 \, d\theta,
$$

we get

$$
r = \left(\frac{\pi R_0^2 - |\Omega_h|}{\pi R_0^2 - |\Omega_g|}\right)^{\frac{1}{2}} < 1.
$$

Clearly $\Omega_g^r = \Omega_{g_r}$ for $g_r(\theta) = rg(\theta)$. Define the function $v_r : \Omega_{g_r} \to \mathbb{R}^2$ as

$$v_r(z) = \begin{cases} u_0\left(\frac{z}{|z|}R_0\right) & \text{if } rR_0 \leq |z| \leq R_0 \\ v\left(\frac{z}{r}\right) & \text{if } g_r\left(\frac{z}{|z|}\right) \leq |z| < rR_0. \end{cases}$$

Since $\Omega_{g_r} \supset \Omega_g$, we see that $|\Omega_{g_r}\Delta\Omega_g| = |\Omega_h| - |\Omega_g|$. Using the inequality (5.3) we have, for Λ sufficiently large, that

$$\mathcal{F}(g_r, v_r) + \Lambda\big||\Omega_{g_r}| - |\Omega_h|\big| + c\sqrt{(|\Omega_{g_r}\Delta\Omega_h| - \varepsilon)^2 + \varepsilon}$$

$$- \mathcal{F}(g, v) - \Lambda\big||\Omega_g| - |\Omega_h|\big| - c\sqrt{(|\Omega_g\Delta\Omega_h| - \varepsilon)^2 + \varepsilon}$$

$$\leq \int_{rR_0 \leq |z| \leq R_0} Q(E(v_r))\,dz - \Lambda\left(|\Omega_h| - |\Omega_g|\right) + c|\Omega_{g_r}\Delta\Omega_g|$$

$$\leq C(1-r) - (\Lambda - 1)\left(|\Omega_h| - |\Omega_g|\right)$$

$$\leq C(|\Omega_h| - |\Omega_g|) - (\Lambda - 1)\left(|\Omega_h| - |\Omega_g|\right) < 0,$$

which contradicts the minimality of (g, v). $\qquad\square$

In the following we study convergence properties of solutions for constrained obstacle problem (5.2).

Lemma 5.6. *Let h be as in Theorem 5.4. Assume $g_n \in BV_\#(\mathbb{R})$ is such that $g_n \leq h + 1/n$ and it satisfies the uniform exterior ball condition. If*

(5.5) $$g_n \to h \text{ in } L^1 \text{ and } \lim_{n\to\infty} \mathcal{H}^1(\Gamma_{g_n} \cup \Sigma_{g_n}) = \mathcal{H}^1(\Gamma_h),$$

then $g_n \to h$ in L^∞. Moreover, for n sufficiently large, g_n are uniformly Lipschitz continuous.

Proof. Here we follow an argument from [14, Theorem 6.9, Steps 1 and 2]. We claim that

$$\sup_{\mathbb{R}} |g_n - h| \to 0 \text{ as } n \to +\infty.$$

Let us first note that $\Gamma_{g_n} \cup \Sigma_{g_n}$ is a connected compact set. Up to a subsequence, we can assume that $\Gamma_{g_n} \cup \Sigma_{g_n}$ converges in the Hausdorff distance to some compact connected set K. The continuity of h and condition (5.5) imply that $\Gamma_h \subset K$. By Gołąb's semicontinuity Theorem (see, e.g. [3, Theorem 4.4.17]) and assumption (5.5) we see that

$$\mathcal{H}^1(\Gamma_h) \leq \mathcal{H}^1(K) \leq \lim_{n\to+\infty} \mathcal{H}^1(\Gamma_{g_n} \cup \Sigma_{g_n}) = \mathcal{H}^1(\Gamma_h).$$

This implies that $\mathcal{H}^1(K \setminus \Gamma_h) = 0$. Since K is connected, it follows from a density lower bound (see, e.g. [3, Lemma 4.4.5]) that $K = \Gamma_h$. Now the claim follows from the definition of the Hausdorff metric and from the continuity of h.

From the previous claim we see that, for n sufficiently large, it holds $\gamma \leq g_n \leq R_0 - \gamma$ for some $\gamma > 0$ small, independent from n. Hence, since the polar coordinates mapping is a C^∞-local diffeomorphism far from the origin, the representation in polar coordinates of F_{g_n} (still denoted by F_{g_n}) satisfies the uniform exterior ball condition up to changing the radius r to $\tilde{r} \in (0, 1)$ by a factor depending only on γ. Now we prove that g_n are L-Lipschitz with $L \leq \frac{8}{\tilde{r}}\|h\|_{C^1(\mathbb{R})}$.

We argue by contradiction and assume that there exists θ and $\theta_k \to \theta$ such that

$$\lim_{k\to\infty} \frac{|g_n(\theta_k) - g_n(\theta)|}{|\theta_k - \theta|} \geq \frac{8}{\tilde{r}}\|h\|_{C^1(\mathbb{R})}$$

and set $z = (\theta, g_n(\theta))$. Without loss of generality we may assume that $\theta_k > \theta$. By the uniform exterior ball condition we find a ball $B_{\tilde{r}}(z_0) \subset \mathbb{R}^2 \setminus F_{g_n}$ such that $\partial B_{\tilde{r}}(z_0) \cap (\Gamma_{g_n} \cup \Sigma_{g_n}) = \{z\}$ and

$$z_0 = z + \tilde{r}\left(\frac{M}{\sqrt{1 + M^2}}, \frac{1}{\sqrt{1 + M^2}}\right), \quad \text{for} \quad M \geq \frac{4}{\tilde{r}}\|h\|_{C^1(\mathbb{R})}$$

Let $z' \in \partial B_{\tilde{r}}(z_0)$ such that

$$z' = z_0 - \tilde{r} \left(\frac{\sqrt{M^2 - 3}}{\sqrt{1 + M^2}}, \frac{2}{\sqrt{1 + M^2}} \right).$$

We write $z' =: z + \tilde{r}(w_1, w_2)$ with

$$w_1 = \frac{M - \sqrt{M^2 - 3}}{\sqrt{1 + M^2}} > 0 \quad \text{and} \quad w_2 = \frac{-1}{\sqrt{1 + M^2}} < 0$$

and since $B_{\tilde{r}}(z_0) \subset \mathbb{R}^2 \setminus F_{g_n}$ we have $g_n(\theta + \tilde{r}w_1) \leq g_n(\theta) + \tilde{r}w_2$. Setting $\delta_n = \sup_{\mathbb{R}} |h - g_n|$ and recalling $||h||_{C^1(\mathbb{R})} \leq M/4$ we get

$$h(\theta + \tilde{r}w_1) \geq h(\theta) - \frac{\tilde{r}M}{4} w_1 \geq g_n(\theta) - \delta_n - \frac{\tilde{r}M}{4} w_1.$$

Therefore we deduce

$$h(\theta + \tilde{r}w_1) - g_n(\theta + \tilde{r}w_1) \geq -\delta_n - \tilde{r}\left(\frac{M}{4}w_1 - w_2 \right)$$

$$= -\delta_n + \frac{\tilde{r}}{\sqrt{1 + M^2}} \left(1 - \frac{M}{4}\left(M - \sqrt{M^2 - 3} \right) \right)$$

$$= -\delta_n + \frac{\tilde{r}}{\sqrt{1 + M^2}} \left(1 - \frac{3M}{4(M + \sqrt{M^2 - 3})} \right) > -\delta_n$$

which, for sufficiently large n, is a contradiction. $\qquad\square$

In the next lemma we show $C^{1,\alpha}$-regularity of the minimizer for the penalized obstacle problem.

Lemma 5.7. *Let h be as in Theorem 5.4 and $(g_n, v_n) \in X(u_0)$ be any minimizer of the problem*
(5.6)
$$\min \left\{ \mathcal{F}(g, v) + \Lambda \big| |\Omega_g| - |\Omega_h| \big| + c\sqrt{(|\Omega_g \Delta \Omega_h| - \varepsilon_n)^2 + \varepsilon_n} \; : \; (g, v) \in X(u_0) \, , \; g \leq h + \frac{1}{n} \right\},$$

where $c \in [0, 1]$ and $\varepsilon_n \to 0$. Assume also that $g_n \to h$ in L^1 and that

$$\lim_{n \to \infty} \mathcal{H}^1(\Gamma_{g_n} \cup \Sigma_{g_n}) = \mathcal{H}^1(\Gamma_h) \quad \text{and} \quad \sup_n \int_{\Omega_{g_n}} Q(E(v_n)) dz < +\infty.$$

Then for all $\alpha \in \left(0, \frac{1}{2}\right)$ and for n large enough $g_n \in C^{1,\alpha}(\mathbb{R})$, the sequence $\{\nabla v_n\}$ is equibounded in $C^{0,\alpha}(\overline{\Omega}_{g_n}; \mathbb{M}^{2\times 2})$, and $g_n \to h$ in $C^{1,\alpha}(\mathbb{R})$.

Proof. From Lemma 5.6 we infer that g_n is sufficiently regular to ensure a decay estimate for ∇v_n. Indeed, for $z_0 \in \Gamma_{g_n}$ there exist $c_n > 0$, a radius r_n and an exponent $\alpha_n \in (0, 1/2)$ such that

$$\int_{B_r(z_0) \cap \Omega_{g_n}} |\nabla v_n|^2 \leq c_n r^{1 + 2\alpha_n},$$

for every $r < r_n$. This follows from the fact that v_n minimizes the elastic energy in Ω_{g_n} and the boundary Γ_{g_n} is Lipschitz, see Theorem 3.13 in [12].

Since g_n is Lipschitz, we may extend v_n in $B_r(z_0)$ such that

(5.7)
$$\int_{B_r(z_0)} |\nabla \tilde{v}_n|^2 \leq c_n r^{1 + 2\alpha_n},$$

where $\tilde{v}_n$ stands for the extension.

For $r < r_n$, denote by z_r' and z_r'' the two points on $\Gamma_{g_n} \cap \partial B_r(z_0)$ such that the open sub-arcs of Γ_{g_n} with end points z_r', z_0 and z_r'', z_0 are contained in $\Gamma_{g_n} \cap \partial B_r(z_0)$. Setting $z_r' = g_n(\theta_r')\sigma(\theta_r')$ and $z_r'' = g_n(\theta_r'')\sigma(\theta_r'')$, denote by l the line segment joining z_r' and z_r'' and define

$$\tilde{g}_n(\theta) := \begin{cases} g_n(\theta) & \theta \in [0, 2\pi) \setminus (\theta_r', \theta_r'') \\ \min\{h(\theta) + \frac{1}{n}, l(\theta)\} & \theta \in (\theta_r', \theta_r''), \end{cases}$$

where $l(\theta)$ is the polar representation of l.

By (5.7) and by the minimality of the pair (g_n, v_n) we have

$$(5.8) \qquad \mathcal{H}^1(\Gamma_{g_n} \cap B_r(z_0)) - \mathcal{H}^1(\Gamma_{\tilde{g}_n} \cap B_r(z_0)) \leq C_n r^{1+2\alpha_n}.$$

Indeed we can estimate

$$\begin{aligned}
0 \geq{}& \mathcal{F}(g_n, v_n) - \mathcal{F}(\tilde{g}_n, \tilde{v}_n) + \Lambda \left(\big| |\Omega_{g_n}| - |\Omega_h| \big| - \big| |\Omega_{\tilde{g}_n}| - |\Omega_h| \big| \right) \\
&+ c \left(\sqrt{(|\Omega_{g_n} \Delta \Omega_h| - \varepsilon_n)^2 + \varepsilon_n} - \sqrt{(|\Omega_{\tilde{g}_n} \Delta \Omega_h| - \varepsilon_n)^2 + \varepsilon_n} \right) \\
\geq{}& \mathcal{H}^1(\Gamma_{g_n} \cap B_r(z_0)) - \mathcal{H}^1(\Gamma_{\tilde{g}_n} \cap B_r(z_0)) - \int_{B_r(z_0)} Q(E(\tilde{v}_n)) dz - (\Lambda + 1)\pi r^2 \\
\geq{}& \mathcal{H}^1(\Gamma_{g_n} \cap B_r(z_0)) - \mathcal{H}^1(\Gamma_{\tilde{g}_n} \cap B_r(z_0)) - C_n r^{1+2\alpha_n}
\end{aligned}$$

We will show later that

$$(5.9) \qquad \mathcal{H}^1(\Gamma_{\tilde{g}_n} \cap B_r(z_0)) - \mathcal{H}^1(l) \leq Cr^2.$$

Now the inequality (5.9) together with (5.8) gives us

$$(5.10) \qquad \mathcal{H}^1(\Gamma_{g_n} \cap B_r(z_0)) - \mathcal{H}^1(l) \leq Cr^{1+2\alpha_n}$$

and the desired $C^{1,\alpha}$-regularity follows from a classical result for quasiminimizers of area functional (see Theorem 1 in [20]) once we observe that
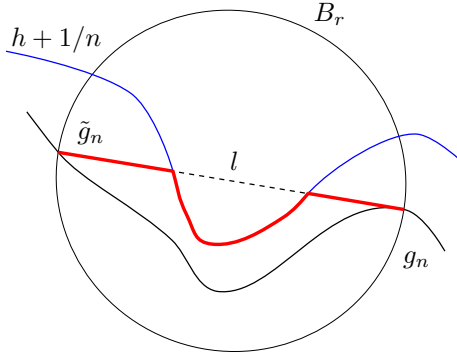
$$\mathcal{H}^1(l) = \inf \left\{ P(F; B_r(z_0)) \ : \ F\Delta\Omega_{g_n} \Subset B_r(z_0) \right\}.$$

The proof of (5.9) is a consequence of the C^2-regularity of h and goes as follows (see Figure 5.2):

$$\begin{aligned}
\mathcal{H}^1(\Gamma_{\tilde{g}_n} \cap B_r(z_0)) - \mathcal{H}^1(l) &\leq \int_{\theta_r'}^{\theta_r''} \sqrt{(\tilde{g}_n(\theta))^2 + (\tilde{g}_n'(\theta))^2} - \sqrt{(l(\theta))^2 + (l'(\theta))^2} \, d\theta \\
&\leq \frac{1}{\gamma} \int_{\theta_r'}^{\theta_r''} (\tilde{g}_n^2 - l^2) \, d\theta + \frac{1}{\gamma} \int_{\theta_r'}^{\theta_r''} (\tilde{g}_n' + l')(\tilde{g}_n' - l') \, d\theta \\
&\leq \frac{1}{\gamma} |B_r(z_0)| + \frac{C}{\gamma} \int_{\theta_r'}^{\theta_r''} (\tilde{g}_n' - l') \, d\theta,
\end{aligned}$$

where C depends on the Lipschitz norm of $\tilde{g}_n$ and l in the interval (θ_r', θ_r'') and γ is a positive constant with $\gamma < \min_{\mathbb{R}} h$.

FIGURE 5.2.

To estimate the last term we first note that either the set $\{h + 1/n < l\}$ is empty or there exists $\theta_0 \in (\theta'_r, \theta''_r)$ such that $\tilde{g}'_n(\theta_0) - l'(\theta_0) = 0$ and using second order Taylor expansion around θ_0 we easily get

$$\int_{\theta'_r}^{\theta''_r} (\tilde{g}'_n - l') \, d\theta \le C r^2$$

where C depends on the C^2-norm of h.

Now we claim that g_n converges to h in C^1-norm. As in the proof of Lemma 5.6 we will work in the plane (θ, ρ) and we recall that the subgraph of g_n, still denoted by F_{g_n}, satisfies the uniform exterior ball condition. From the C^1-regularity and the uniform Lipschitz estimate, in the Lemma 5.6, we obtain $\sup_n \|g_n\|_{C^1} < \infty$. Hence, from the uniform exterior ball condition we conclude that at every point there exists a parabola touching g_n from above. In other words, there is $C > 0$ such that for every θ_0 it holds for $P(\theta) = g_n(\theta_0) + g'_n(\theta_0)(\theta - \theta_0) + C(\theta - \theta_0)^2$ that

$$\min_\theta (P - g_n) = (P - g_n)(\theta_0) = 0.$$

This implies that g_n are uniformly semiconcave, i.e., for every n the function

$$\theta \mapsto g_n(\theta) - C \theta^2$$

is convex. We may now use the Lemma 5.3 to conclude the desired C^1-convergence of g_n.

The convergence of g_n to h in C^1-norm allows us to use a blow-up method (see [14, Theorem 6.10]) to infer the uniform estimate

(5.11)
$$\int_{B_r(z_0)} |\nabla v_n|^2 \le c_0 r^{1+2\sigma}$$

for any $\sigma \in (1/2, 1)$ and for all $r < r_0$ where c_0 and r_0 are independent of n.

Once we have (5.11), we can repeat the argument used to prove (5.10), replacing (5.7) by (5.11), to infer

$$\mathcal{H}^1(\Gamma_{g_n} \cap B_r(z_0)) - \mathcal{H}^1(l) \le C r^{1+2\sigma}.$$

This implies a uniform estimate for the $C^{1,\alpha}$-norms of g_n for $\alpha \in (0, 1/2)$ (see for instance [9, Proposition 2.2]). The $C^{1,\alpha}$-convergence of g_n now follows by a compactness argument.

To conclude the proof we have just to observe that, since v_n is a solution of the Lamé system in Ω_{g_n}, we can apply the elliptic estimates provided in [14, Proposition 8.9] to deduce that ∇v_n is uniformly bounded in $C^{0,\alpha}(\overline{\Omega}_{g_n}, \mathbb{R}^2 \times \mathbb{R}^2)$ for all $\alpha \in (0, 1/2)$. □

Lemma 5.8. *Let* $(h, u) \in X_{\mathrm{reg}}(u_0)$ *be a critical point of* $\mathcal{F}$ *such that* $0 < h < R_0$, *and let* (g_n, v_n) *be as in the previous lemma. Suppose that* $\nabla v_n \rightharpoonup \nabla u$ *weakly in* $L^2_{\mathrm{loc}}(\Omega_h; \mathbb{R}^2 \times \mathbb{R}^2)$,

$$\lim_{n \to \infty} \int_{\Omega_{g_n}} Q(E(v_n)) dz = \int_{\Omega_h} Q(E(u)) dz \quad and \quad \lim_{n \to \infty} \frac{|\Omega_{g_n} \Delta \Omega_h|}{\varepsilon_n} = 1.$$

Then $g_n \in C^{1,1}(\mathbb{R})$ *and* $g_n \to h$ *in* $C^{1,1}(\mathbb{R})$, *for* n *sufficiently large.*

Proof. From Lemma 5.7 we know that $g_n \to h$ in $C^{1,\alpha}(\mathbb{R})$. Therefore for large n there exist diffeomorphisms $\Phi_n : \overline{\Omega}_{g_n} \to \overline{\Omega}_h$ such that $\Phi_n \to id$ in $C^{1,\alpha}$. Let B_R be any ball of radius $R \in (R_0 - \max_{\mathbb{R}} h, R_0)$. Since

$$\sup_{n \in \mathbb{N}} \left\{ \|v_n\|_{C^{1,\alpha}(\overline{\Omega}_{g_n})} \right\} < \infty$$

by the convergence $\nabla v_n \rightharpoonup \nabla u$ we have that

(5.12)
$$\nabla v_n \circ \Phi_n^{-1} \to \nabla u \qquad \text{in } C^{0,\alpha}(\overline{\Omega}_h \cap B_R; \mathbb{M}^{2\times 2}).$$

To prove the claim denote $I_n := \{\theta \in [0, 2\pi] \mid g_n(\theta) < h(\theta) + 1/n =: h_n(\theta)\}$. Since I_n is open, we may write $I_n = \bigcup_{i=1}^\infty (a_i^n, b_i^n)$. Notice that

(5.13)
$$g'_n(\theta) = h'_n(\theta) = h'(\theta) \qquad \text{on } [0, 2\pi] \setminus I_n.$$

If I_n is empty, the claim is trivial. Therefore we may assume that $I_n \ne \emptyset$. Since $g_n \in C^{1,\alpha}(\mathbb{R})$, we can write the Euler-Lagrange equation for (g_n, v_n) in the weak sense:

(5.14)
$$k_{g_n}(\theta) = Q(E(v_n))(\theta, g_n(\theta)) + \beta_n(\theta, g_n(\theta)) + \lambda_n, \qquad \theta \in I_n.$$

Here

$$\beta_n = \frac{\Lambda \, |\Omega_{g_n} \Delta \Omega_h|}{\sqrt{(|\Omega_{g_n} \Delta \Omega_h| - \varepsilon_n)^2 + \varepsilon_n}} \, \mathrm{sign} \, (\chi_{\Omega_h} - \chi_{\Omega_{g_n}})$$

and λ_n is some Lagrange multiplier. Notice that from the assumption $\lim_{n \to \infty} \varepsilon_n^{-1} |\Omega_{g_n} \Delta \Omega_h| = 1$ it follows that

$$(5.15) \qquad |\beta_n| = \frac{\Lambda \, |\Omega_{g_n} \Delta \Omega_h|}{\sqrt{(|\Omega_{g_n} \Delta \Omega_h| - \varepsilon_n)^2 + \varepsilon_n}} \le \Lambda \, \frac{|\Omega_{g_n} \Delta \Omega_h|}{\sqrt{\varepsilon_n}} \to 0.$$

Recall the Euler-Lagrange equation for (h, u)

$$(5.16) \qquad\qquad k_h(\theta) = Q(E(u))(\theta, h(\theta)) + \lambda_\infty \, .$$

We will show that $\lambda_n \to \lambda_\infty$. Notice that for the curvature in polar coordinates it holds

$$k_{g_n} \, g_n = \frac{g_n^2 + 2g_n'^2 - g_n g_n''}{(g_n^2 + g_n'^2)^{\frac{3}{2}}} \, g_n = -\left(\frac{g_n'}{\sqrt{g_n^2 + g_n'^2}} \right)' + \frac{g_n}{\sqrt{g_n^2 + g_n'^2}}.$$

Hence, multiplying (5.14) by g_n, integrating over I_n and using (5.16) yield

$$\int_{I_n} \left[Q(E(v_n)\big(\theta, g_n(\theta)\big) + \beta_n\big(\theta, g_n(\theta)\big) + \lambda_n \right] g_n \, d\theta = \int_{I_n} k_{g_n} \, g_n \, d\theta$$

$$= \int_{I_n} -\left(\frac{g_n'}{\sqrt{g_n^2 + g_n'^2}} \right)' + \frac{g_n}{\sqrt{g_n^2 + g_n'^2}} \, d\theta$$

$$= \sum_{i=1}^\infty -\left(\frac{g_n'(b_i^n)}{\sqrt{g_n^2(b_i^n) + g_n'^2(b_i^n)}} - \frac{g_n'(a_i^n)}{\sqrt{g_n^2(a_i^n) + g_n'^2(a_i^n)}} \right) + \int_{a_i^n}^{b_i^n} \frac{g_n}{\sqrt{g_n^2 + g_n'^2}} \, d\theta$$

$$= \sum_{i=1}^\infty -\left(\frac{h_n'(b_i^n)}{\sqrt{h_n^2(b_i^n) + h_n'^2(b_i^n)}} - \frac{h_n'(a_i^n)}{\sqrt{h_n^2(a_i^n) + h_n'^2(a_i^n)}} \right) + \int_{a_i^n}^{b_i^n} \frac{g_n}{\sqrt{g_n^2 + g_n'^2}} \, d\theta$$

$$= \int_{I_n} k_{h_n} \, h_n \, d\theta + \int_{I_n} \frac{g_n}{\sqrt{g_n^2 + g_n'^2}} - \frac{h_n}{\sqrt{h_n^2 + h_n'^2}} \, d\theta$$

$$= \int_{I_n} \left[Q(E(u)\big(\theta, h(\theta)\big) + \lambda_\infty \right] h \, d\theta + \int_{I_n} (k_{h_n} \, h_n - k_h \, h) + \frac{g_n}{\sqrt{g_n^2 + g_n'^2}} - \frac{h_n}{\sqrt{h_n^2 + h_n'^2}} \, d\theta.$$

Recall that $h_n = h + 1/n$. Therefore by (5.12), (5.15) and the previous calculations we conclude that

$$\lim_{n \to \infty} \frac{1}{|I_n|} \int_{I_n} \lambda_n g_n - \lambda_\infty h \, d\theta = 0 \, ,$$

which clearly implies $\lambda_n \to \lambda_\infty$.

From (5.13) and (5.14) we conclude that $g_n \in C^{1,1}(\mathbb{R})$. Moreover by the equations (5.13), (5.14) and (5.16) together with the convergences (5.12), (5.15) and $\lambda_n \to \lambda_\infty$ we conclude that

$$k_{g_n} \to k_h \qquad \text{in } L^\infty.$$

This in turn gives us the convergence

$$g_n'' \to h'' \qquad \text{in } L^\infty.$$

$$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \square$$

Now we are in position to prove the main theorem of this section.

Proof of Theorem 5.1. **Step 1:** We show first that (h, u) is a strict local minimizer, i.e., we prove the claim without the estimate on the right-hand side of (5.1).

Observe that from the results of the previous section we may assume that (h, u) is a $C^{1,1}$-local minimizer. The result will follow once we prove that $C^{1,1}$-local minimality implies the local

minimality. Arguing by contradiction let us assume that for any $n \in \mathbb{N}$ there exist $(h_n, u_n) \in X(u_0)$ with $|\Omega_{h_n}| = |\Omega_h|$ such that

$$\mathcal{F}(h_n, u_n) \leq \mathcal{F}(h, u) \quad \text{and} \quad 0 < d_{\mathcal{H}}(\Gamma_{h_n} \cup \Sigma_{g_n}, \Gamma_h) \leq \frac{1}{n}.$$

Consider the sequence $(g_n, v_n) \in X(u_0)$ of minimizers of the following penalized obstacle problem

$$\min \left\{ \mathcal{F}(g, v) + \Lambda \big| |\Omega_g| - |\Omega_h| \big| \; : \; (g, v) \in X(u_0), \, g \leq h + \frac{1}{n} \right\},$$

for some large Λ. Since (h_n, u_n) and (h, u) are clearly competitors, we have that

$$\mathcal{F}(g_n, v_n) \leq \mathcal{F}(h_n, u_n) \leq \mathcal{F}(h, u).$$

By the contradiction assumption we may assume that $(h_n, u_n) \neq (h, u)$.

By the compactness property of $X(u_0)$ there exists (g, v) such that, up to subsequences, $(g_n, v_n) \to (g, v)$ in $X(u_0)$. Let $(f, w) \in X(u_0)$ with $f \leq h$, by the lower semicontinuity of $\mathcal{F}$ and the minimality of (g_n, v_n), we get

(5.17)
$$\mathcal{F}(g, v) + \Lambda \big| |\Omega_g| - |\Omega_h| \big| \leq \liminf_{n \to \infty} \left[\mathcal{F}(g_n, v_n) + \Lambda \big| |\Omega_{g_n}| - |\Omega_h| \big| \right]$$
$$\leq \mathcal{F}(f, w) + \Lambda \big| |\Omega_f| - |\Omega_h| \big|.$$

Choosing $(f, w) = (h, v)$ in the previous inequality, we obtain that

(5.18)
$$\mathcal{H}^1(\Gamma_g) + \Lambda \big| |\Omega_g| - |\Omega_h| \big| \leq \mathcal{H}^1(\Gamma_h)$$

When Λ is sufficiently large, (5.18) and Lemma 5.2 imply that $g = h$. Moreover, we observe that from (5.17) it follows that (h, v) minimizes $\mathcal{F}$ in the class of all $(f, w) \in X(u_0)$ with $f \leq h$. In particular v must coincide with the elastic equilibrium u.

Choosing $(f, w) = (h, u)$ in (5.17), we deduce

$$\lim_{n \to \infty} \mathcal{H}^1(\Gamma_{g_n} \cup \Sigma_{g_n}) = \mathcal{H}^1(\Gamma_h),$$

$$\lim_{n \to \infty} \int_{\Omega_{g_n}} Q(E(v_n)) \, dz = \int_{\Omega_h} Q(E(u)) \, dz.$$

From Lemma 5.8 we get $g_n \to h$ in $C^{1,1}(\mathbb{R})$.

We only need to modify g_n such that it satisfies the volume constraint. We simply define $\tilde{g}_n(\theta) := g_n(\theta) + \delta_n$ where δ_n are chosen so that $|\Omega_{\tilde{g}_n}| = |\Omega_h|$. By Lemma 5.5 it holds $|\Omega_{g_n}| \geq |\Omega_h|$ and therefore $\delta_n \geq 0$ and $\Omega_{\tilde{g}_n} \subset \Omega_{g_n}$. Hence v_n is well defined in $\Omega_{\tilde{g}_n}$ and $(\tilde{g}_n, v_n)$ is an admissible pair.

Since $h > 0$ and $g_n \to h$ uniformly, we have $g_n > \gamma$ for some $\gamma > 0$ independent from n and $\delta_n \to 0$. We may estimate

$$\mathcal{H}^1(\Gamma_{\tilde{g}_n}) - \mathcal{H}^1(\Gamma_{g_n}) = \int_0^{2\pi} \sqrt{(g_n + \delta_n)^2 + g_n'^2} - \sqrt{g_n^2 + g_n'^2} \, d\theta$$

$$\leq \frac{1}{\gamma} \int_0^{2\pi} 2 g_n \delta_n + \delta_n^2 \, d\theta$$

and

$$\big| |\Omega_{\tilde{g}_n}| - |\Omega_{g_n}| \big| = \frac{1}{2} \int_0^{2\pi} (g_n + \delta_n)^2 - g_n^2 \, d\theta = \frac{1}{2} \int_0^{2\pi} 2 g_n \delta_n + \delta_n^2 \, d\theta.$$

Therefore whenever $\Lambda \geq \frac{2}{\gamma}$ we have

(5.19)
$$\mathcal{H}^1(\Gamma_{\tilde{g}_n}) - \mathcal{H}^1(\Gamma_{g_n}) \leq \Lambda \big| |\Omega_{\tilde{g}_n}| - |\Omega_{g_n}| \big|.$$

The claim now follows, since by the choice of $\tilde{g}_n$ and by (5.19) we have

$$\mathcal{F}(\tilde{g}_n, v_n) = \mathcal{F}(\tilde{g}_n, v_n) + \Lambda \big| |\Omega_{\tilde{g}_n}| - |\Omega_h| \big|$$
$$\leq \mathcal{F}(g_n, v_n) + \Lambda \big| |\Omega_{g_n}| - |\Omega_h| \big| \leq \mathcal{F}(h_n, u_n) \leq \mathcal{F}(h, u).$$

This contradicts the fact that (h, u) is a strict $C^{1,1}$-local minimizer.

Step 2: We will now prove the theorem. The proof is very similar to the first step. Arguing by contradiction we assume that there are $(h_n, u_n) \in X(u_0)$ with $|\Omega_{h_n}| = |\Omega_h|$ such that

$$\mathcal{F}(h_n, u_n) \leq \mathcal{F}(h, u) + c_0 |\Omega_{h_n} \Delta \Omega_h|^2 \quad \text{and} \quad 0 < d_{\mathcal{H}}(\Gamma_{h_n} \cup \Sigma_{g_n}, \Gamma_h) \leq \frac{1}{n}.$$

Denote $\varepsilon_n := |\Omega_{h_n} \Delta \Omega_h|$. Notice that $d_{\mathcal{H}}(\Gamma_{h_n} \cup \Sigma_{g_n}, \Gamma_h) \to 0$ implies $\chi_{\Omega_{h_n}} \to \chi_{\Omega_h}$ in L^1 and therefore $\varepsilon_n \to 0$.

This time we replace the contradicting sequence (h_n, u_n) by $(g_n, v_n) \in X(u_0)$ which minimizes

$$\min \left\{ \mathcal{F}(g, v) + \Lambda \big||\Omega_g| - |\Omega_h|\big| + \sqrt{(|\Omega_g \Delta \Omega_h| - \varepsilon_n)^2 + \varepsilon_n} \; : \; (g, v) \in X(u_0),\, g \leq h + \frac{1}{n} \right\}.$$

By compactness we may assume that, up to a subsequence, $(g_n, v_n) \to (g, v)$ in $X(u_0)$. By completely similar argument as in Step 1 we conclude that $(g, v) = (h, u)$ whenever Λ is sufficiently large. Moreover, we have that

$$\lim_{n \to \infty} \mathcal{H}^1(\Gamma_{g_n} \cup \Sigma_{g_n}) = \mathcal{H}^1(\Gamma_h),$$

$$\lim_{n \to \infty} \int_{\Omega_{g_n}} Q(E(v_n))\, dz = \int_{\Omega_h} Q(E(u))\, dz.$$

To conclude that $g_n \to h$ in $C^{1,1}(\mathbb{R})$, it remains to show that

$$(5.20) \qquad \lim_{n \to \infty} \frac{|\Omega_{g_n} \Delta \Omega_h|}{\varepsilon_n} = 1$$

and apply Lemma 5.8.

Suppose that (5.20) were false. Then there exists $c > 0$ such that $\big||\Omega_{g_n} \Delta \Omega_h| - \varepsilon_n\big| \geq c\varepsilon_n$. Use the minimality of (g_n, v_n) and the contradiction assumption for (h_n, u_n), we obtain

$$(5.21) \qquad \begin{aligned} \mathcal{F}(g_n, v_n) &+ \Lambda \big||\Omega_{g_n}| - |\Omega_h|\big| + \sqrt{(|\Omega_{g_n} \Delta \Omega_h| - \varepsilon_n)^2 + \varepsilon_n} \\ &\leq \mathcal{F}(h_n, u_n) + \sqrt{\varepsilon_n} \\ &< \mathcal{F}(h, u) + c_0 \varepsilon_n^2 + \sqrt{\varepsilon_n}. \end{aligned}$$

Now we observe that from [13, Proposition 6.1], for Λ sufficiently large, (h, u) is also a minimizer of the penalized problem

$$\mathcal{F}(g, v) + \Lambda \big||\Omega_g| - |\Omega_h|\big|.$$

Hence we have

$$(5.22) \qquad \mathcal{F}(h, u) \leq \mathcal{F}(g_n, v_n) + \Lambda \big||\Omega_{g_n}| - |\Omega_h|\big|.$$

Combining (5.21) and (5.22) we get

$$\sqrt{c^2 \varepsilon_n^2 + \varepsilon_n} \leq \sqrt{(|\Omega_{g_n} \Delta \Omega_h| - \varepsilon_n)^2 + \varepsilon_n} < c_0\, \varepsilon_n^2 + \sqrt{\varepsilon_n},$$

which is a contradiction since $\varepsilon_n \to 0$ proving (5.20).

Arguing as in (5.21) and by using (5.20) we obtain

$$(5.23) \qquad \begin{aligned} \mathcal{F}(g_n, v_n) + \Lambda \big||\Omega_{g_n}| - |\Omega_h|\big| &\leq \mathcal{F}(h_n, u_n) + \sqrt{\varepsilon_n} - \sqrt{(|\Omega_{g_n} \Delta \Omega_h| - \varepsilon_n)^2 + \varepsilon_n} \\ &< \mathcal{F}(h, u) + c_0 \varepsilon_n^2 \\ &\leq \mathcal{F}(h, u) + 2c_0 |\Omega_{g_n} \Delta \Omega_h|^2, \end{aligned}$$

when n is large.

As in Step 1 define $\tilde{g}_n(\theta) := g_n(\theta) + \delta_n$ where $\delta_n \geq 0$ are such that $|\Omega_{\tilde{g}_n}| = |\Omega_h|$. By choosing Λ large enough we have

$$(5.24) \qquad \mathcal{H}^1(\Gamma_{\tilde{g}_n}) - \mathcal{H}^1(\Gamma_{g_n}) \leq \frac{\Lambda}{2} \big||\Omega_{\tilde{g}_n}| - |\Omega_{g_n}|\big|.$$

Therefore since

$$|\Omega_{g_n} \Delta \Omega_h|^2 \leq 2|\Omega_{\tilde{g}_n} \Delta \Omega_h|^2 + 2|\Omega_{\tilde{g}_n} \Delta \Omega_{g_n}|^2 = 2|\Omega_{\tilde{g}_n} \Delta \Omega_h|^2 + 2\big||\Omega_{g_n}| - |\Omega_h|\big|^2$$

we have by (5.23) and (5.24) that

$$
\begin{aligned}
\mathcal{F}(\tilde{g}_n, v_n) &\leq \mathcal{F}(g_n, v_n) + \frac{\Lambda}{2} \big| |\Omega_{g_n}| - |\Omega_h| \big| \\
&< \mathcal{F}(h, u) + 2c_0 \, |\Omega_{g_n} \Delta \Omega_h|^2 - \frac{\Lambda}{2} \big| |\Omega_{g_n}| - |\Omega_h| \big| \\
&\leq \mathcal{F}(h, u) + 4c_0 \, |\Omega_{\tilde{g}_n} \Delta \Omega_h|^2 - \frac{\Lambda}{2} \big| |\Omega_{g_n}| - |\Omega_h| \big| + 4c_0 \big| |\Omega_{g_n}| - |\Omega_h| \big|^2 \\
&\leq \mathcal{F}(h, u) + 4c_0 \, |\Omega_{\tilde{g}_n} \Delta \Omega_h|^2,
\end{aligned}
$$

when n is sufficiently large. This contradicts Proposition 4.1 when c_0 is chosen to be small enough. $\qquad\square$

6. The case of the disk

In this section we consider the particular case when a radial stretching is applied to a material with round cavity $F = \bar{B}_r$. We prove that the disk remains stable under small radial stretching. This result is similar to the case of flat configuration in [14]. The main difference to the flat case, where the minimal shape is a rectangle, is that the curvature of the disk is nonzero and therefore the second variation formula becomes considerably more complicated. Instead of trying to explicitly write the second variation, we use fine estimates to find a range of stability.

The Dirichlet boundary condition has the form of radial stretching,

$$
(6.1) \qquad u_0\big(\rho \sigma(\theta)\big) = \alpha R_0 \sigma(\theta) \qquad \text{for } \rho \geq R_0 \,,
$$

where $\alpha \in \mathbb{R}$ is some constant. The region occupied by the elastic material is the annulus $A(R_0, r) := B_{R_0} \setminus \bar{B}_r$. For u_0 as above we say that $(h, u) \in X(u_0)$ is a *round configuration* if $h(\theta) \equiv r$ and u is the elastic equilibrium associated to h.

For the next theorem we define

$$
\beta(t) := 1 + \frac{\mu + \lambda}{\mu} \frac{t^2}{R_0^2} \,.
$$

Recall also the definition of the ellipticity constant $\eta = \min\{\mu, \mu + \lambda\}$.

Theorem 6.1. *Let*

$$
r_0 := \sup \left\{ t \leq R_0 \mid (1 + t^2) \log\left(\frac{R_0}{t}\right) \geq \frac{\eta}{4\mu} \right\}
$$

and define the function $G : \mathbb{R} \to [-\infty, R_0)$ *as*

$$
G(\alpha) := \sup \left\{ t \leq R_0 \mid t \log\left(\frac{R_0}{t}\right) \beta^2(t) \geq \frac{\eta}{32(\mu + \lambda)^2 \alpha^2} \right\} \,.
$$

If $r \in (r_0, R_0)$ *and* $\alpha \in \mathbb{R}$ *satisfy*

$$
(6.2) \qquad r > G(\alpha) \,,
$$

then the round configuration is a strict local minimizer of $\mathcal{F}$ *under the volume constraint.*

The elastic equilibrium u can be explicitly calculated. Indeed, because of the symmetry we can write

$$
u(\rho \sigma(\theta)) = f(\rho) \sigma(\theta)
$$

and applying the first equation in (2.7) we have

$$
f''(\rho) + \frac{f'(\rho)}{\rho} - \frac{f(\rho)}{\rho^2} = 0 \,.
$$

This can be easily solved

$$
f(\rho) = \frac{a}{\rho} + b\rho,
$$

for some $a, b \in \mathbb{R}$. To find a and b observe that

(6.3)
$$\mathbb{C}E(u) = 2\mu \begin{pmatrix} f'(\rho)\cos^2\theta + \frac{f(\rho)}{\rho}\sin^2\theta & \left(f'(\rho) - \frac{f(\rho)}{\rho}\right)\sin\theta\cos\theta \\ \left(f'(\rho) - \frac{f(\rho)}{\rho}\right)\sin\theta\cos\theta & f'(\rho)\sin^2\theta + \frac{f(\rho)}{\rho}\cos^2\theta \end{pmatrix} + \lambda\left(f'(\rho) + \frac{f(\rho)}{\rho}\right)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Therefore, the second equation in (2.7) gives

$$(2\mu + \lambda)f'(r) + \lambda\frac{f(r)}{r} = 0.$$

This and the Dirichlet condition (6.1) yield

(6.4)
$$\frac{a}{r^2} = \frac{\mu + \lambda}{\mu}b \quad \text{and} \quad b = \frac{\alpha}{\beta(r)}.$$

It is trivial to check that the round configuration is a critical point of $\mathcal{F}$. To prove the Theorem 6.1 we need to show that the round configuration is a point of positive second variation. To this aim, let us explicitly write the quadratic form (3.16). By (6.3) and (6.4) we have

$$\mathbb{C}E(u) = 4b(\mu + \lambda)\begin{pmatrix} \sin^2\theta & -\sin\theta\cos\theta \\ -\sin\theta\cos\theta & \cos^2\theta \end{pmatrix} = 4b(\mu + \lambda)\ \tau \otimes \tau,$$

on the boundary ∂B_r. Hence, for $\psi \in H^1_\#(\mathbb{R})$, we have

$$\operatorname{div}_\tau(\langle\underline{\psi}, \nu\rangle\mathbb{C}E(u)) = 4b(\mu + \lambda)(-\langle\underline{\psi}, \nu\rangle\nu + \partial_\tau\langle\underline{\psi}, \nu\rangle\tau)$$

and the equation (3.17) for u_ψ becomes

(6.5)
$$\int_{A(R,r)} \mathbb{C}E(u_\psi) : E(w)\,dz = -4b(\mu + \lambda)\int_{\partial B_r} \left(-\langle\underline{\psi}, \nu\rangle\langle w, \nu\rangle + (\partial_\tau\langle\underline{\psi}, \nu\rangle)\langle w, \tau\rangle\right)\,d\mathcal{H}^1.$$

Moreover, in the case of a round configuration the elastic energy is

(6.6)
$$Q(E(u)) = 2(\mu + \lambda)b^2 + 2\mu\frac{a^2}{\rho^4}$$

and therefore, by (6.4), we get

$$\partial_\nu Q(E(u)) = -\frac{8(\mu + \lambda)^2}{\mu}\frac{b^2}{r} \quad \text{on} \quad \partial B_r.$$

Hence, (3.16) becomes

(6.7)
$$\partial^2\mathcal{F}(h, u)[\psi] = -\int_{A(R,r)} 2Q(E(u_\psi))\,dz + \int_{\partial B_r} |\partial_\tau\langle\underline{\psi}, \nu\rangle|^2\,d\mathcal{H}^1$$
$$+ \int_{\partial B_r} \left(\frac{8(\mu + \lambda)^2}{\mu}\frac{b^2}{r} - \frac{1}{r^2}\right)\langle\underline{\psi}, \nu\rangle^2\,d\mathcal{H}^1,$$

where $u_\psi \in \mathcal{A}(B_R \setminus \bar{B}_r)$ solves (6.5), and ψ satisfies $\int_0^{2\pi} \psi\,d\theta = 0$.

Now the goal of is to prove that (6.7) is positive whenever the assumptions of Theorem 6.1 are satisfied. The main obstacle is to bound the first term in (6.7) which will be done by using the equation (6.5). To this aim we need the following simple lemma, which we prove to keep track of the optimal constant.

Lemma 6.2. *Suppose that $v \in W^{1,2}(A(R_0, r); \mathbb{R}^2)$ is a continuous map with $v = 0$ on ∂B_{R_0} and A is a matrix. Then for $w(z) = v(z) + Az$ we have that*

$$\int_{\partial B_r} |w|^2\,d\mathcal{H}^1 \leq r\log\left(\frac{R_0}{r}\right)\int_{A(R_0,r)} \left|Dv - \frac{r}{R_0 - r}A\right|^2\,dz.$$

Proof. Consider w in polar coordinates. Fix an angle θ and integrate over $[r, R_0]$

$$AR_0\sigma(\theta) - w(r\sigma(\theta)) = \int_r^{R_0} Dw\,(\rho\sigma(\theta))\,\sigma(\theta)\,d\rho,$$

which implies

$$|w(r\sigma(\theta))| \leq \int_r^{R_0} \left| Dv(\sigma(\theta)) - \frac{r}{R_0 - r}A \right| d\rho.$$

Integrate over θ and use Hölder's inequality to obtain

$$\int_0^{2\pi} |w(\rho\sigma(\theta))|^2\,d\theta \leq \int_0^{2\pi} \left(\int_r^{R_0} \left| Dv(\rho\sigma(\theta)) - \frac{r}{R_0 - r}A \right| d\rho \right)^2 d\theta$$

$$\leq \int_0^{2\pi} \left(\int_r^{R_0} \frac{1}{\rho}\,d\rho \cdot \int_r^{R_0} \left| Dv(\rho\sigma(\theta)) - \frac{r}{R_0 - r}A \right|^2 \rho\,d\rho \right) d\theta$$

$$= \log\left(\frac{R_0}{r}\right) \int_{A(R_0,r)} \left| Dv - \frac{r}{R_0 - r}A \right|^2 dz.$$

The inequality follows from $\int_{\partial B_r} |w|^2\,d\mathcal{H}^1 = r\int_0^{2\pi} |w(r,\theta)|^2\,d\theta$. $\qquad\square$

Proof of Theorem 6.1. As we stated before, by the local minimality criterion it is enough to prove that the second variation of $\mathcal{F}$ at (h, u) is positive. Suppose that $\psi \in H^1_\#(\mathbb{R})$ satisfies $\int_0^{2\pi} \psi\,d\theta = 0$ and $\psi \neq 0$. Without loss of generality we may assume ψ to be smooth. To estimate the first term in (6.7) we claim that

$$(6.8) \qquad 2\int_{A(R_0,r)} Q(E(u_\psi))\,dz \leq \frac{32(\mu + \lambda)^2 b^2}{\eta} r\log\left(\frac{R_0}{r}\right) \int_{\partial B_r} \langle\underline{\psi}, \nu\rangle^2 + |\partial_\tau\langle\underline{\psi}, \nu\rangle|^2\,d\mathcal{H}^1.$$

To this aim, choose $w(z) = u_\psi(z) + Az$ as a test function in (6.5) where A is antisymmetric, to obtain

$$(6.9) \quad 2\int_{A(R_0,r)} Q(E(u_\psi))\,dz = -4b(\mu + \lambda)\int_{\partial B_r} \left(-\langle\underline{\psi}, \nu\rangle\langle w, \nu\rangle + \partial_\tau\langle\underline{\psi}, \nu\rangle\langle w, \tau\rangle\right)\,d\mathcal{H}^1$$

$$\leq 4b(\mu + \lambda)\left(\int_{\partial B_r} \langle\underline{\psi}, \nu\rangle^2 + |\partial_\tau\langle\underline{\psi}, \nu\rangle|^2\,d\mathcal{H}^1\right)^{1/2}\left(\int_{\partial B_r} |w|^2\,d\mathcal{H}^1\right)^{1/2}.$$

Apply Lemma 6.2 to w to get

$$(6.10) \qquad \int_{\partial B_r} |w|^2\,d\mathcal{H}^1 \leq r\log\left(\frac{R_0}{r}\right) \int_{A(R_0,r)} \left| Du_\psi - \frac{r}{R - r}A \right|^2 dz.$$

Let $R_k \to \infty$ and for every k choose an antisymmetric A_k such that

$$\int_{A(R_k,r)} Du_\psi - \frac{r}{R - r}A_k\,dz = \int_{A(R_k,r)} Du_\psi^T + \frac{r}{R - r}A_k\,dz.$$

By Theorem 2.5 we get

$$\int_{A(R_k,r)} \left| Du_\psi - \frac{r}{R - r}A_k \right|^2 dz \leq C_k \int_{A(R_k,r)} |E(u_\psi)|^2\,dz = C_k \int_{A(R,r)} |E(u_\psi)|^2\,dz.$$

Together with (6.10) this yields

$$\int_{\partial B_r} |w|^2\,d\mathcal{H}^1 \leq r\log\left(\frac{R_0}{r}\right) C_k \int_{A(R_0,r)} |E(u_\psi)|^2\,dz.$$

Since $C_k \to 4$ as $R_k \to \infty$ we have that

$$\int_{\partial B_r} |w|^2\,d\mathcal{H}^1 \leq \frac{4r}{\eta}\log\left(\frac{R_0}{r}\right) \int_{A(R_0,r)} Q(E(u_\psi))\,dz.$$

Now (6.8) follows from (6.9) and from the previous inequality.

We estimate (6.7) by using (6.8) and obtain

$$\partial^2 \mathcal{F}(h,u)[\psi] \geq -32\eta^{-1}(\mu+\lambda)^2 b^2 \, r \log\left(\frac{R_0}{r}\right) \int_{\partial B_r} \langle \underline{\psi}, \nu \rangle^2 + |\partial_\tau \langle \underline{\psi}, \nu \rangle|^2 \, d\mathcal{H}^1$$

$$+ \int_{\partial B_r} |\partial_\tau \langle \underline{\psi}, \nu \rangle|^2 \, d\mathcal{H}^1 + \int_{\partial B_r} \left(\frac{8(\mu+\lambda)^2}{\mu}\frac{b^2}{r} - \frac{1}{r^2}\right) \langle \underline{\psi}, \nu \rangle^2 \, d\mathcal{H}^1$$

$$(6.11) \qquad = \int_{\partial B_r} |\partial_\tau \langle \underline{\psi}, \nu \rangle|^2 - \frac{1}{r^2} \langle \underline{\psi}, \nu \rangle^2 \, d\mathcal{H}^1$$

$$- 32\eta^{-1}(\mu+\lambda)^2 b^2 \, r \log\left(\frac{R_0}{r}\right) \int_{\partial B_r} |\partial_\tau \langle \underline{\psi}, \nu \rangle|^2 \, d\mathcal{H}^1$$

$$+ \left(\frac{r}{\mu} - 4\eta^{-1} r^3 \log\left(\frac{R_0}{r}\right)\right) 8(\mu+\lambda)^2 b^2 \int_{\partial B_r} \frac{1}{r^2} \langle \underline{\psi}, \nu \rangle^2 \, d\mathcal{H}^1.$$

Let us first treat the last term in (6.11). For every $r > r_0$ we have that

$$\partial^2 \mathcal{F}(h,u)[\psi] > \left(1 - 32\eta^{-1}(\mu+\lambda)^2 b^2 \, r \log\left(\frac{R_0}{r}\right)\right) \int_{\partial B_r} |\partial_\tau \langle \underline{\psi}, \nu \rangle|^2 - \frac{1}{r^2} \langle \underline{\psi}, \nu \rangle^2 \, d\mathcal{H}^1.$$

Furthermore, if (6.2) is satisfied, then

$$1 - 32\eta^{-1}(\mu+\lambda)^2 b^2 \, r \log\left(\frac{R_0}{r}\right) > 0 \,.$$

By the definition (3.1) we see that $\langle \underline{\psi}, \nu \rangle = \psi\left(\sigma^{-1}\left(\frac{z}{|z|}\right)\right)$. Hence, by the Wirtinger's inequality, we get

$$\int_{\partial B_r} |\partial_\tau \langle \underline{\psi}, \nu \rangle|^2 - \frac{1}{r^2} \langle \underline{\psi}, \nu \rangle^2 \, d\mathcal{H}^1 = \frac{1}{r} \int_0^{2\pi} |\psi'(\theta)|^2 - |\psi(\theta)|^2 \, d\theta \geq 0 \,.$$

which concludes the proof. $\square$

At the end of the section we study the global minimality of the round configuration. We begin with the following remark.

Remark 6.3. Suppose that R_0 and r_0 are as in Theorem 6.1 and fix $\alpha \in \mathbb{R}$ and a small $\varepsilon > 0$. Then for every $r \in [r_0 + \varepsilon, R_0]$ such that $r \geq G(\alpha) + \varepsilon$ the proof above actually gives

$$\partial^2 \mathcal{F}(h,u)[\psi] \geq c_2 \int_0^{2\pi} |\psi'(\theta)|^2 - |\psi(\theta)|^2 \, d\theta + c_1 \int_0^{2\pi} |\psi(\theta)|^2 \, d\theta,$$

for some small $0 < c_1 < c_2$, independent of r. Using the Wirtinger's inequality we get

$$\partial^2 \mathcal{F}(h,u)[\psi] \geq c_0 ||\psi||^2_{H^1([0,2\pi))},$$

for c_0 depending only on R_0, r_0, α and ε. This is a uniform version of the Lemma 3.6.

We can use this uniform bound of the constant c_0 to prove a uniform local $C^{1,1}$-minimality of the round configuration for $r \in [r_0 + \varepsilon, R_0]$ with $r \geq G(\alpha) + \varepsilon$. Indeed, arguing as in the Proposition 4.1 and in the Lemma 4.7 we conclude that there is $\delta > 0$ such that for any $(g, v) \in X(u_0)$ with $|F_g| = |B_r|$ and $||g - r||_{C^{1,1}(\mathbb{R})} \leq \delta$ it holds

$$\mathcal{F}(g, v) \geq \mathcal{F}(r, u_r),$$

where u_r stands for the elastic equilibrium associated to the disk B_r.

The previous remark enables us to prove the global minimality of the disk when the volume of the annulus is small.

Proposition 6.4. *Suppose that R_0 is the radius of the large ball and u_0 is the Dirichlet boundary conditions as in (6.1) with fixed $\alpha > 0$. There exists $r_{glob} < R_0$ such that for every $r \in (r_{glob}, R_0)$ the round configuration, with a disk B_r, is a global minimizer of $\mathcal{F}$ under the volume constraint.*

Proof. We argue by contradiction and assume that there exist a sequence of radii $r_n \nearrow R_0$ and a sequence $(k_n, w_n) \in X(u_0)$ of minimizers of $\mathcal{F}$ under the volume constraint $|\Omega_{k_n}| = |A(R_0, r_n)|$ such that

$$\mathcal{F}(k_n, w_n) < \mathcal{F}(r_n, u_n),$$

where u_n stands for the elastic equilibrium relative to r_n. Since (k_n, w_n) minimizes $\mathcal{F}$ we immediately have that $\mathcal{H}^1(\Gamma_{k_n} \cup \Sigma_{k_n}) \to 2\pi R_0$. Therefore, since F_{k_n} is connected, we deduce that $\varepsilon_n := d_{\mathcal{H}}(\Gamma_{k_n} \cup \Sigma_{k_n}, \Gamma_{r_n}) \to 0$ as $n \to \infty$.

We may calculate the elastic equilibrium

$$u_n(\rho, \theta) = \left(\frac{a_n}{\rho} + b_n \rho \right) \sigma(\theta),$$

where

$$b_n = \left(1 + \frac{\mu + \lambda}{\mu} \frac{r_n^2}{R_0^2} \right)^{-1} \alpha \quad \text{and} \quad a_n = \frac{\mu + \lambda}{\mu} r_n^2 \, b_n.$$

By the Remark 6.3 we have that for large n it holds

$$\partial^2 \mathcal{F}(r_n, u_n)[\psi] \geq c_0 ||\psi||^2_{H^1(\partial B_{r_n})},$$

for $\int_0^{2\pi} \psi \, d\theta = 0$, where c_0 is independent of n.

We note that u_n is also the elastic equilibrium in the annulus $A(R, r_n)$, for any $R > R_0$, with respect to its own boundary conditions on ∂B_R, $v(R, \theta) = u_n(R, \theta)$. For $R > R_0$ we define

$$\mathcal{F}_R(g, v) = \int_{B_R \setminus F_g} Q(E(v)) \, dz + \mathcal{H}^1(\Gamma_g) + 2\mathcal{H}^1(\Sigma_g)$$

and

$$X_R(u_n) = \{ (g, v) \mid g \in BV_\#(\mathbb{R}), \, v \in H^1_{\text{loc}}(\mathbb{R}^2 \setminus F_g; \mathbb{R}^2), \, v = u_n \text{ outside } B_R \}.$$

Consider the estimate (6.11) for $\partial^2 \mathcal{F}_{R_1}(r_n, u_n)[\psi]$, i.e., replace R_0, r and b by R_1, r_n and b_n. By continuity we may choose R_1 close to R_0 such that

$$\partial^2 \mathcal{F}_{R_1}(r_n, u_n)[\psi] \geq \frac{c_0}{2} ||\psi||^2_{H^1(\partial B_{r_n})},$$

for $\int_0^{2\pi} \psi \, d\theta = 0$. Arguing as in the Remark 6.3 we conclude that (r_n, u_n) is a local $C^{1,1}$-minimizer of $\mathcal{F}_{R_1}$ uniformly in n, i.e., there is $\delta > 0$, independent of n, such that for any $(g, v) \in X_{R_1}(u_n)$, with $||g - r_n||_{C^{1,1}(\mathbb{R})} < \delta$, it holds

$$(6.12) \qquad\qquad \mathcal{F}_{R_1}(g, v) \geq \mathcal{F}_{R_1}(r_n, u_n).$$

Define

$$\tilde{w}_n(z) := \begin{cases} w_n(z) & \text{if } z \in \bar{B}_{R_0} \setminus F_{k_n} \\ u_n(z) & \text{if } z \in A(R_1, R_0). \end{cases}$$

By the assumption on (k_n, w_n) it holds

$$(6.13) \qquad\qquad \mathcal{F}_{R_1}(k_n, \tilde{w}_n) < \mathcal{F}_{R_1}(r_n, u_n).$$

Suppose that (g_n, v_n) is a solution of the problem

$$\min\{\mathcal{F}_{R_1}(g, v) + \Lambda \big| |F_g| - |B_{r_n}| \big| : (g, v) \in X_{R_1}(u_n), \, g \leq r_n + \varepsilon_n\},$$

where Λ is large. Arguing as in Lemma 5.6, Lemma 5.7 and Lemma 5.8 we conclude that $g_n \to R_0$ in $C^{1,1}(\mathbb{R})$. In particular, $||g_n - r_n||_{C^{1,1}(\mathbb{R})} \to 0$.

By the minimality of (g_n, v_n) we have that $\mathcal{F}_{R_1}(g_n, v_n) + \Lambda \big| |F_g| - |B_{r_n}| \big| \leq \mathcal{F}_{R_1}(k_n, \tilde{w}_n)$. Defining $\tilde{g}_n = g_n + \delta_n$ such that $|F_{\tilde{g}_n}| = |B_{r_n}|$ we obtain, as in (5.19), that

$$(6.14) \qquad \mathcal{F}_{R_1}(\tilde{g}_n, v_n) \leq \mathcal{F}_{R_1}(g_n, v_n) + \Lambda \big| |F_g| - |B_{r_n}| \big| \leq \mathcal{F}_{R_1}(k_n, \tilde{w}_n),$$

when Λ is large enough. Moreover $\delta_n \to 0$. Hence $||\tilde{g}_n - r_n||_{C^{1,1}(\mathbb{R})} \to 0$ and therefore (6.12), (6.13) and (6.14) imply

$$\mathcal{F}_{R_1}(r_n, u_n) \leq \mathcal{F}_{R_1}(\tilde{g}_n, v_n) \leq \mathcal{F}_{R_1}(k_n, \tilde{w}_n) < \mathcal{F}_{R_1}(r_n, u_n),$$

which is a contradiction.

$\square$

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